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Monotonicity of Primal-Dual Interior-Point Algorithms for Semidefinite Programming Problems⋆

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Abstract: We present primal-dual interior-point algorithms with polynomial iteration bounds to find approximate solutions of semidefinite programming problems. Our algorithms achieve the current best iteration bounds and, in every iteration of our algorithms, primal and dual objective values are strictly improved.

Key words: Semidefinite Programming, Interior-Point Methods, Primal-Dual Algorithms, Monotonicity of Objective Values, Search Directions, Polynomial Complexity

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1 Introduction

Many of the interior-point algorithms, which have polynomial iteration bounds, can be seen as algorithms balancing two criteria:

- Centering,
- the true objective.

As a result, as soon as the polynomial-time interior-point algorithms for linear programming (LP) emerged, the question of whether the current best polynomial bound and strict monotonicity in the true objective at every iteration can be achieved by an interior-point algorithm was asked. Anstreicher [1] answered the question in the affirmative for Karmarkar's algorithm (also see the related paper [2]). Monteiro, Adler and Resende [20] showed that their primal-dual affine-scaling algorithm achieves polynomial iteration bound $O(n \log^2(1/\epsilon))$ while being strictly monotone in both primal and dual objective functions in each iteration. (Here, n denotes the larger of the dimensions defining the LP problem, and $0 < \epsilon < 1$ denotes the desired ratio of the final duality gap to the initial duality gap.) Freund and Tan [7] obtained a strictly monotone, $O(n \log(1/\epsilon))$ iteration algorithm for LP as a by-product of their work on parametric centers for LP. Mizuno and Nagasawa [16] showed that Todd's low complexity algorithm [26] can be modified so that it is strictly monotone in the original objective, while keeping the same worst-case iteration bound $O(\sqrt{n} \log(1/\epsilon))$.

We are interested in the question: "Is it possible to achieve the current best iteration bound for Semidefinite Programming (SDP) in a primal-dual interior-point algorithm which strictly improves primal and dual objective values in every iteration ? " From now on, we refer to the latter mentioned property (strict improvement in both objective values in every iteration) as *strict monotonicity*.

Answer to the above question, in the case of LP is affirmative. Mizuno, Todd and Tunçel [17] showed that by a careful choice of the centering parameter in each iteration, it is possible to keep the polynomial iteration bounds of primal-dual path-following algorithm the same, while achieving strict monotonicity.

In this paper, we utilize a description of the Nesterov-Todd direction [23, 24, 27] to express the primal-dual search directions in a scaled space as orthogonal projections of *scaled centering* and *scaled affine-scaling* components. Then, using the elementary properties of projections, we extend the analysis of [17] to the SDP setting. We also investigate the related properties of search directions derived from the general, local, conditions of Güler and Tunçel [11].

The reasons to consider strict monotonicity in the case of SDP are essentially the same as those suggested in [17]. However, we feel that some of these reasons may be magnified in some applications of the SDP techniques as we elaborate below.

- In practice, the step size is usually determined as a "fixed fraction of the way to the boundary of the semidefinite cone." The next iterate is not well-defined if one of the search directions is a recession direction of the underlying feasible region.
- If different step sizes are used in the primal and the dual spaces then the duality gap may increase from one iteration to the next.

In the case of SDP problems, the amount of computation time spent on step size calculations seems to be relatively (relative to the amount of time spent on computing the search direction) more significant than in the case of LP. Therefore, the above concerns seem even more important in the case of SDP than LP.

Another reason of interest in strictly monotone algorithms comes from applications. In many popular applications (e.g., in combinatorial optimization [9], control and system theory [29], structural design [4], robust optimization [5]), the SDP problem being solved is **only an approximation** of the real problem, and in many cases we only obtain bounds from the underlying SDP (and perhaps some guidance for heuristics to construct feasible solutions as well—see for instance Goemans and Williamson [10], Benson, Ye and Zhang [3]). In such applications, the problems are usually very large and one can only afford a few hands full of iterations. It would certainly be annoying if in such an application the objective value that gives the desired bound gets worse from an iteration to the next.

In the next section, we present our notation, components of the search directions, and some general results about the strict monotonicity of a family of primal-dual interior-point algorithms. These results apply to a very general class convex optimization problems (those in so-called *conic form*, including SDP problems). Moreover, even for the LP problems, our elementary arguments establish the strict monotonicity of a large family of search directions for the first time. In Section 3, we focus on primal-dual interior-point algorithms derived using the Nesterov-Todd scaling. Our analyses of the algorithms are similar to the analyses of Monteiro [18], Monteiro and Zhang [21] and Sturm and Zhang [25]. However, our analyses seem even closer to the LP case, namely those of Kojima, Mizuno and Yoshise [14], Kojima, Megiddo, Noma and Yoshise [13] and in particular to the analysis given in [17].

2 Notation, Definitions and Basic Results

We consider the semidefinite programming(SDP) problems in the following primal (P) and dual (D) forms.

$$\begin{array}{lll} (P) \quad \min & \langle C, X\rangle & \\ & \mathcal{A}(X) & = \quad b, \\ & X & \in \quad \Sigma^n_+, \end{array}$$

$$\begin{array}{lll} (D) \quad \max & b^T y & \\ & \mathcal{A}^*(y) + S & = \quad C, \\ & S & \in \quad \Sigma^n_+, \end{array}$$

where Σ^n is the space of $n \times n$ symmetric matrices over the reals, Σ^n_+ is the cone of $n \times n$ symmetric positive semidefinite matrices, and $\mathcal{A}$ is a linear operator from Σ^n to R^m. We denote by Σ^n_{++} the cone of $n \times n$ symmetric positive definite matrices. The inner-product in (P) is the trace inner-product, that is $\langle C, X\rangle = \mathrm{tr}(C^T X)$. $\mathcal{A}^*$ denotes the adjoint of $\mathcal{A}$. Without loss of generality, we assume that $\mathcal{A}$ is surjective. If not, we can consider the representation $A_i \bullet X = b_i, i \in \{1, \ldots, m\}$, where $A_i \in \mathcal{S}^n$ for every i. $\mathcal{A}$ being surjective is equivalent to $A_1, A_2, \ldots, A_m$ being linearly independent. The latter can be assumed without loss of generality, since if they are linearly dependent, then either the system $A_i \bullet X = b_i, i \in \{1, \ldots, m\}$ has no solution, or there are some redundant equations which can be eliminated. In the first case, (P) is infeasible. In the second case, all redundant equations, and corresponding A_i, b_i can be eliminated, to arrive at an equivalent problem satisfying the assumption. Under this assumption, for any solution, (y, S), of the equation $\sum_{i=1}^{m} y_i A_i + S = C$, the S part of the solution uniquely identifies the corresponding y. Sometimes, in interior-point algorithms, it is convenient to refer only to S when one mentions a feasible solution of (D).

We assume throughout this paper that there exists

$$(X^{(0)}, y^{(0)}, S^{(0)}) \in \Sigma^n_{++} \oplus R^m \oplus \Sigma^n_{++}$$

such that

$$\mathcal{A}(X^{(0)}) = b, \quad \mathcal{A}^*(y^{(0)}) + S^{(0)} = C,$$

and that the algorithms we consider, are provided with such a starting point. (Under these assumptions, primal and dual optimal values are equal and they are attained, see, e.g., [22]). We will confine ourselves to path-following algorithms using various neighborhoods of the central

path. Therefore, individual algorithms will have the assumption that such a starting point lies in the corresponding neighborhood of the central path.

Let us recall the well-known weak duality relation. Note that for any feasible solution (X, y, S) we have

$$\langle C, X\rangle - b^T y = \langle X, S\rangle \geq 0.$$

Therefore, if (X, y, S) is feasible and $\langle X, S\rangle = 0$ then X is optimal in (P) and (y, S) is optimal in (D). (Under our assumptions about $(X^{(0)}, y^{(0)}, S^{(0)})$, the converse is also true.)

For a symmetric matrix A, $\lambda_{\min}(A)$ denotes the smallest eigenvalue of A. $\lambda_{\max}(A)$ denotes the maximum absolute eigenvalue of A. Sometimes we write $\lambda_{\min}(AB)$, where A and B are symmetric and A or B is positive definite. Say A is positive definite, then AB and $A^{1/2}BA^{1/2}$ are similar and hence have the same eigenvalues. We use this fact repeatedly. We have the following specializations of a well-known inequality

$$\lambda_{\max}(AB) \;\leq\; \frac{1}{2}\lambda_{\max}\left(AB + BA\right), \tag{1}$$

$$\lambda_{\min}(AB) \;\geq\; \frac{1}{2}\lambda_{\min}\left(AB + BA\right). \tag{2}$$

For these relations, Horn and Johnson [12] credit Mirsky. For the usage of such inequalities in the iteration complexity analyses of interior-point algorithms, see Monteiro [18].

Many search directions for SDP problems have been proposed so far. For a review and a unified iteration complexity analyses of such directions, see Kojima, Shida and Shindoh [15]. Our study is centered around the following family of search directions.

Definition 2.1 *(Güler and Tunçel [11], see [28]) Given $X, S \in \Sigma^n_{++}$, a linear transformation T on the space of $n \times n$ symmetric matrices is said to be in $\mathcal{T}(X, S)$ if*

(a) T is symmetric ($T^* = T$) and nonsingular,

(b) $T(S) = T^{-1}(X) =: V$,

(c) $T(X^{-1}) = T^{-1}(S^{-1}) = V^{-1}$,

(d) $\lambda_{\min}(V) = \left(\lambda_{\min}(X^{1/2}SX^{1/2})\right)^{1/2}$.

Once we have such a T, we can define the data of the underlying *scaled* primal-dual pair as follows. First we define $\bar{\mathcal{A}} := \mathcal{A}\left(T(\cdot)\right)$. Then

$$\bar{d}_x := T^{-1}(d_x), \;\; \bar{d}_s := T(d_s),$$

where d_x, d_s denote the search directions in the original primal and dual spaces respectively. We also define $\mu := \langle X, S\rangle/n$ and $U := V^{-1} - \gamma\mu V$. Throughout this paper, whenever we write μ, we mean $\langle X, S\rangle/n$ for the pair (X, S) which will be clear from the context. Note that $\langle V, V\rangle = \langle X, S\rangle = n\mu$. Now, we consider the system of linear equations:

$$\begin{aligned} \bar{\mathcal{A}}(\bar{d}_x) &= 0 \quad (3)\\ \bar{\mathcal{A}}^*(\bar{d}_y) + \bar{d}_s &= 0. \quad (4)\\ \bar{d}_x + \bar{d}_s &= U. \quad (5) \end{aligned}$$

Clearly, $(\bar{d}_x, \bar{d}_s)$ solves the above system iff

$$\bar{d}_x = \mathcal{P}_{\bar{\mathcal{A}}}(U),$$

and

$$\bar{d}_s = U - \mathcal{P}_{\bar{\mathcal{A}}}(U).$$

That is, $\bar{d}_x$ and $\bar{d}_s$ are orthogonal projections of U onto the null space of $\bar{\mathcal{A}}$ and the range of $\bar{\mathcal{A}}^*$ respectively. When $\gamma = 0$ in the definition of U, we call the resulting search direction the *primal-dual affine-scaling* direction, since in case of LP, the search direction of [20] is a special case. We remark that even in the case of LP problems, we obtain many "affine-scaling" directions due to the flexibility in the choice of $T \in \mathcal{T}(X, S)$. We also define the following matrices and conventions of notation:

$$W := \frac{1}{\sqrt{\mu}} V,$$

$$Z := \sqrt{\mu} V^{-1}.$$

Note that

$$U = \sqrt{\mu}(\gamma Z - W).$$

For every $B \in \Sigma^n$,

$$B_p := \mathcal{P}_{\bar{\mathcal{A}}}(B) \quad \text{and} \quad B_q := B - \mathcal{P}_{\bar{\mathcal{A}}}(B).$$

Under these definitions, we have

$$\bar{d}_x = \mathcal{P}_{\bar{\mathcal{A}}}(\gamma\mu V^{-1} - V) = \sqrt{\mu}(\gamma Z_p - W_p)$$

and

$$\bar{d}_s = \sqrt{\mu}(\gamma Z_q - W_q).$$

In *every* inner-product space, we have the following elementary and well-known properties of the projections. Let $\|\cdot\|_F$ denote the norm induced by $\langle\cdot,\cdot\rangle$ (that is, $\|A\|_F = \sqrt{\mathrm{tr}(A^T A)}$, the Frobenius norm); then for all $A, B \in \Sigma^n$ and projections $\mathcal{P}_{\bar{\mathcal{A}}}$ the following hold:

$$\langle A, B_p\rangle = \langle A_p, B\rangle = \langle A_p, B_p\rangle \quad (6)$$

$$\langle A_p, B_q \rangle = 0 \tag{7}$$

$$\|A_p\|_F^2 + \|A_q\|_F^2 = \|A\|_F^2 \tag{8}$$

$$\|A\|_F^2 = 0 \ \text{ iff } \ A = 0. \tag{9}$$

Using the above construction of the search directions and the elementary properties of the projections we have the following theorem.

Theorem 2.1 *All primal-dual affine-scaling algorithms are strictly monotone in both primal and dual objectives, unless all the primal or all the dual solutions are optimal.*

Proof. Let (X, y, S) denote the current iterate. Also let $T \in \mathcal{T}(X, S)$. Note that

$$\begin{aligned} \mathcal{A}^*(y) + S = C \quad &\Longleftrightarrow \quad T(\mathcal{A}^*(y)) + T(S) = T(C) \\ &\Longleftrightarrow \quad V_p = (T(C))_p. \end{aligned}$$

Now we compute the change in the primal objective value along the search direction d_x.

$$\begin{array}{rcccl} \langle C, d_x \rangle & = & \langle C, T(\bar{d}_x) \rangle & = & \langle T(C), \bar{d}_x \rangle \\ & = & -\sqrt{\mu}\langle T(C), W_p \rangle & = & -\sqrt{\mu}\langle (T(C))_p, W_p \rangle \\ & = & -\sqrt{\mu}\langle V_p, W_p \rangle & = & -\mu \|W_p\|_F^2 \qquad \leq \quad 0. \end{array}$$

We have "=" above iff $W_p = 0$. But, $W_p = 0$ implies $(T(C))_p = 0$. Thus $\hat{S} := 0$ is feasible in (D). Therefore, all feasible solutions of (P) are optimal.

Similarly, we compute the change in the dual objective value along the search direction (d_y, d_s).

$$\begin{array}{rcccl} b^T d_y & = & (\mathcal{A}(X))^T d_y & = & \langle X, \mathcal{A}^*(d_y) \rangle \\ & = & -\langle X, d_s \rangle & = & -\langle T^{-1}(X), \bar{d}_s \rangle \\ & = & \sqrt{\mu}\langle V, W_q \rangle & = & \mu \|W_q\|_F^2 \qquad \geq \quad 0. \end{array}$$

We have "=" above iff $W_q = 0$. But, $W_q = 0$ implies $0 = \bar{\mathcal{A}}(V) = \mathcal{A}(X) = b$. Thus $\hat{X} := 0$ is feasible in (P). Therefore, all feasible solutions of (D) are optimal. □

Remark 2.1 *The above theorem uses only the properties (a) and (b) of Definition 2.1. Using only these two conditions, we see that the notion of primal-dual affine-scaling directions can be extended to general conic convex optimization setting (replace Σ_+^n by K, a pointed closed convex cone in R^n with nonempty interior). The above theorem, with the same proof (with any suitable choice of the inner-product) extends to this general conic convex optimization setting.*

Remark 2.2 *de Klerk, Roos and Terlaky [6] showed that $O(n\log^2(1/\epsilon))$ iteration bound applies to the primal-dual affine-scaling algorithm (with suitably short steps, as in the LP case) derived using the Nesterov-Todd scaling. Combining this result with our Theorem 2.1, we see that for this special primal-dual affine-scaling algorithm for SDP problems, the current best polynomial iteration bound and strict monotonicity are simultaneously achieved.*

For the rest of the paper, we assume that $b \neq 0$ and that there does not exist $y \in R^m$ such that $\mathcal{A}^*(y) = C$. This is equivalent to saying that the primal objective function is not constant and the dual objective function value is not constant on respective feasible regions. This is also equivalent to

$$0 < \|W_p\|_F^2 < n.$$

Note that

$$\|W_p\|_F^2 + \|W_q\|_F^2 = \|W\|_F^2 = n,$$

and

$$n = \langle W, Z\rangle = \langle W_p, Z_p\rangle + \langle W_q, Z_q\rangle.$$

Consider the system of linear equations (3)-(5). Let us denote the unique solution of the system by $(\bar{d}_x(\gamma), \bar{d}_y(\gamma), \bar{d}_s(\gamma))$ when $U = \gamma\mu Z - W$, to make the dependence of the search direction on γ very explicit in the notation. The search direction in the original (unscaled) space is denoted by $(d_x(\gamma), d_y(\gamma), d_s(\gamma))$.

We compute

$$\langle C, d_x(\gamma)\rangle = \langle V_p, \bar{d}_x(\gamma)\rangle = \langle \sqrt{\mu}W_p, \sqrt{\mu}\gamma Z_p - \sqrt{\mu}W_p\rangle = \mu\left(\gamma\langle W_p, Z_p\rangle - \|W_p\|_F^2\right), \tag{10}$$

and similarly for the dual

$$b^T d_y = -\mu\left(\gamma\langle W_q, Z_q\rangle - \|W_q\|_F^2\right). \tag{11}$$

Based on the above computations we can define critical values γ_P and γ_D of the centering parameter for the primal and dual objective functions respectively.

$$\gamma_P := \frac{\|W_p\|_F^2}{\langle W_p, Z_p\rangle}, \quad \gamma_D := \frac{\|W_q\|_F^2}{\langle W_q, Z_q\rangle}$$

in the above definitions, we have $:= +\infty$, if the corresponding denominator is zero.

Lemma 2.1 *Under our assumptions, exactly one of the following cases holds:*

(i) $\gamma_P = \gamma_D = 1$

(ii) $\gamma_P \in (0,1)$ *and* $\gamma_D \in (-\infty, 0) \cup (1, +\infty]$

(iii) $\gamma_P \in (-\infty, 0) \cup (1, +\infty]$ *and* $\gamma_D \in (0, 1)$.

Proof. Note that γ_P cannot equal zero, since this would imply $\|W_p\|_F = 0$; similarly, the possibility of $\gamma_D = 0$ is also eliminated. Thus, the above cases are mutually exclusive and exhaustive for γ_P (and by symmetry for γ_D). Hence, it suffices to show that "$\gamma_P \in (0, 1)$ implies $\gamma_D \in (-\infty, 0) \cup (1, +\infty]$." Suppose $\gamma_P \in (0, 1)$. Then $\langle W_p, Z_p \rangle > \|W_p\|_F^2$. Since

$$\|W_p\|_F^2 + \|W_q\|_F^2 = n = \langle W_p, Z_p \rangle + \langle W_q, Z_q \rangle,$$

we conclude

$$\|W_q\|_F^2 = n - \|W_p\|_F^2 > n - \langle W_p, Z_p \rangle = \langle W_q, Z_q \rangle.$$

Therefore, $\gamma_D \in (-\infty, 0) \cup (0, +\infty]$. □

We define

$$\bar{\gamma} := \begin{cases} \gamma_P, & \text{if } \gamma_P \in (0, 1]; \\ \gamma_D, & \text{if } \gamma_D \in (0, 1]. \end{cases}$$

If either γ_P or γ_D becomes 1, so does the other. In this case, $\bar{\gamma}$ is set to 1 and, any choice $\gamma \in (0, 1)$ ensures strict monotonicity. We essentially proved the following theorem.

Theorem 2.2 *If in computing the search directions, we choose* $\gamma \in [0, \bar{\gamma})$ *then the underlying primal-dual algorithm is strictly monotone in both objectives.*

Proof. Follows from Lemma 2.1 and Equations (10) and (11). □

The following results, Lemmas 2.2 and 2.3 apply to any $T \in \mathcal{T}(X, S)$. However, as we remark after the proof of Lemma 2.3, the main results of the paper, in Section 3, are based on a specific choice of $T \in \mathcal{T}(X, S)$, namely, the choice which corresponds to the the Nesterov-Todd scaling. (See, [23, 24, 25, 27].)

Lemma 2.2 *If* $\gamma_P < +\infty$ *then* $\|W_p - \gamma_P Z_p\|_F \leq |\gamma_P| \ \|W_p - Z_p\|_2$. *If* $\gamma_D < +\infty$ *then* $\|W_q - \gamma_D Z_q\|_F \leq |\gamma_D| \ \|W_q - Z_q\|_2$.

Proof. By the primal-dual symmetry of our set-up, it suffices to prove the first statement. By the definition of γ_P, we have

$$\langle W_p, W_p - \gamma_P Z_p \rangle = 0.$$

Thus

$$\|W_p - \gamma_P Z_p\|_F^2 = \langle \gamma_P (W_p - Z_p), W_p - \gamma_P Z_p \rangle \leq |\gamma_P| \ \|W_p - Z_p\|_2 \|W_p - \gamma_P Z_p\|_F.$$

If $\|W_p - \gamma_P Z_p\|_F = 0$ then the desired inequality holds; otherwise, dividing both sides of the above by $\|W_p - \gamma_P Z_p\|_F$ gives the desired inequality. □

Lemma 2.3 *Let $X, S \in \Sigma_{++}^n$ and $T \in \mathcal{T}(X, S)$ be given. Then the following relations hold.*
(a) $\langle X + \alpha d_x, S + \alpha d_s \rangle = [1 - \alpha(1-\gamma)]\, n\mu$.
(b) $\|W - Z\| \leq \frac{\sqrt{\mu}}{\lambda_{\min}(V)} \left\| \frac{V^2}{\mu} - I \right\|$, where both norms are either operator 2-norm or the Frobenius norm.
(c) Let $\alpha \in [0,1]$ be such that $(V + \alpha \bar{d}_x)$ and $(V + \alpha \bar{d}_s)$ are both positive definite. Then

$$\left\| \frac{(V + \alpha \bar{d}_x)(V + \alpha \bar{d}_s)}{\mu} - [1 - \alpha(1-\gamma)]\, I \right\|_2 \ \leq \ (1 - \alpha) \left\| \frac{V^2}{\mu} - I \right\|_2 + \frac{\alpha^2}{2\mu} \|\bar{d}_x \bar{d}_s + \bar{d}_s \bar{d}_x\|_2.$$

Proof. (a) We compute

$$\begin{aligned} \langle X + \alpha d_x, S + \alpha d_s \rangle &= \langle X, S \rangle + \alpha \left(\langle X, d_s \rangle + \langle S, d_x \rangle \right) + \alpha^2 \langle d_x, d_s \rangle \\ &= n\mu + \alpha \left(\langle V, T(d_s) \rangle + \langle V, T^{-1}(d_x) \rangle \right) \\ &= n\mu + \alpha \langle V, \gamma \mu V^{-1} - V \rangle \\ &= [1 - \alpha(1-\gamma)]\, n\mu. \end{aligned}$$

The second equation above uses Definition 2.1 and the fact that $\langle d_x, d_s \rangle = 0$. The third equation uses the fact $\langle V, V \rangle = n\mu$.

(b) We have

$$\|W - Z\| = \left\| \frac{V}{\sqrt{\mu}} - \sqrt{\mu} V^{-1} \right\| = \left\| \sqrt{\mu} V^{-1} \left(\frac{V^2}{\mu} - I \right) \right\| \leq \left(\frac{\sqrt{\mu}}{\lambda_{\min}(V)} \right) \left\| \frac{V^2}{\mu} - I \right\|.$$

(c) Using the relation (1), we obtain the following inequality

$$\begin{aligned} & 2 \left\| \frac{(V + \alpha \bar{d}_x)(V + \alpha \bar{d}_s)}{\mu} - [1 - \alpha(1-\gamma)]\, I \right\|_2 \\ & \leq \left\| \frac{(V + \alpha \bar{d}_x)(V + \alpha \bar{d}_s) + (V + \alpha \bar{d}_s)(V + \alpha \bar{d}_x)}{\mu} - 2\,[1 - \alpha(1-\gamma)]\, I \right\|_2 \\ = \left\| 2 \left(\frac{V^2}{\mu} - I \right) + \alpha \left(\frac{\bar{d}_x V + \bar{d}_s V + V \bar{d}_x + V \bar{d}_s}{\mu} + 2(1-\gamma) I \right) + \frac{\alpha^2}{\mu} (\bar{d}_x \bar{d}_s + \bar{d}_s \bar{d}_x) \right\|_2 & . \end{aligned}$$

Note that the scaled search direction $(\bar{d}_x, \bar{d}_s)$ satisfies $\bar{d}_x + \bar{d}_s = \gamma\mu V^{-1} - V$. Multiplying this matrix equation from right by V, we obtain $\bar{d}_x V + \bar{d}_s V = \gamma\mu I - V^2$. Multiplying the original equation by V from the left, we obtain $V\bar{d}_x + V\bar{d}_s = \gamma\mu I - V^2$. Adding each side of the last two equations, we conclude

$$\left(\frac{\bar{d}_x V + \bar{d}_s V + V\bar{d}_x + V\bar{d}_s}{\mu}\right) = 2\left(\gamma I - \frac{V^2}{\mu}\right).$$

Therefore,

$$\begin{aligned} 2\left\|\frac{(V+\alpha\bar{d}_x)(V+\alpha\bar{d}_s)}{\mu} - [1-\alpha(1-\gamma)]\,I\right\|_2 &\leq \left\|2\left(\frac{V^2}{\mu} - I\right) + 2\alpha\left(I - \frac{V^2}{\mu}\right) + \frac{\alpha^2}{\mu}(\bar{d}_x\bar{d}_s + \bar{d}_s\bar{d}_x)\right\|_2 \\ &= \left\|2(1-\alpha)\left(\frac{V^2}{\mu} - I\right) + \frac{\alpha^2}{\mu}(\bar{d}_x\bar{d}_s + \bar{d}_s\bar{d}_x)\right\|_2 \\ &\leq 2(1-\alpha)\left\|\frac{V^2}{\mu} - I\right\|_2 + \frac{\alpha^2}{\mu}\|\bar{d}_x\bar{d}_s + \bar{d}_s\bar{d}_x\|_2. \end{aligned}$$

□

Theorems 2.1 and 2.2, and Lemmas 2.1, 2.2, 2.3 apply to a very general class of search directions and are included in this section as they are fundamental. For the rest of the paper, we focus on, one very special $T \in \mathcal{T}(X,S)$ and the search directions arising from it. The next remark provides some merits of doing so. The fundamental results given by Theorem 2.2, Lemmas 2.1, 2.2 and 2.3, as well as Lemma 2.4 are utilized in Section 3.

Remark 2.3 *We may expect that $\left\|\frac{V^2}{\mu} - I\right\|$ can serve as a measure of proximity for the central path. However, notice in the above lemma that the result only implies a relationship in the $V-$space. Once we take a step in the $V-$space we should be able to translate our gain and the proximity to the central path to the original (unscaled) space. One of the ways of ensuring this is to restrict T such that all the eigenvalues of $X^{1/2}SX^{1/2}$ and V^2 are the same, and the measure of proximity applied to the eigenvalues of $(V+\alpha\bar{d}_x)(V+\alpha\bar{d}_s)$ overestimates the measure of proximity of the next iterate, determined by the eigenvalues of $T(V+\alpha\bar{d}_x)T^{-1}(V+\alpha\bar{d}_s)$. The search direction proposed by Nesterov and Todd satisfy all these properties (see Lemma 2.1 of Monteiro [19]). From now on, we restrict our attention to the Nesterov-Todd direction. Whenever we refer to V, it is the unique V arising from the primal-dual pair (X,S) via the Nesterov-Todd scaling.*

Lemma 2.4 *Let $X, S \in \Sigma_{++}^n$. Also let $\alpha \in [0,1]$ be such that $(V+\alpha\bar{d}_x)$ and $(V+\alpha\bar{d}_s)$ are both positive definite. Then*

$$\left\|\frac{(V+\alpha\bar{d}_x)(V+\alpha\bar{d}_s)}{\mu} - [1-\alpha(1-\gamma)]\,I\right\|_F \leq (1-\alpha)\left\|\frac{V^2}{\mu} - I\right\|_F + \frac{\alpha^2}{2\mu}\|\bar{d}_x\bar{d}_s + \bar{d}_s\bar{d}_x\|_F.$$

Proof. Using Lemma 3.3 of Monteiro [18], we obtain the following inequality

$$2\left\|\frac{(V+\alpha\bar{d}_x)(V+\alpha\bar{d}_s)}{\mu}-[1-\alpha(1-\gamma)]\,I\right\|_F$$

$$\leq\left\|\frac{(V+\alpha\bar{d}_x)(V+\alpha\bar{d}_s)+(V+\alpha\bar{d}_s)(V+\alpha\bar{d}_x)}{\mu}-2\,[1-\alpha(1-\gamma)]\,I\right\|_F$$

$$=\left\|2\left(\frac{V^2}{\mu}-I\right)+\alpha\left(\frac{\bar{d}_xV+\bar{d}_sV+V\bar{d}_x+V\bar{d}_s}{\mu}+2(1-\gamma)I\right)+\frac{\alpha^2}{\mu}(\bar{d}_x\bar{d}_s+\bar{d}_s\bar{d}_x)\right\|_F.$$

The rest of the proof is essentially the same as the proof of Lemma 2.3 (c). □

3 Polynomial Iteration Bounds for Strictly Monotone Algorithms

Let us define the set of primal-dual pair of interior solutions.

$$\mathcal{F}_0:=\{(X,S)\in\Sigma^n_{++}\oplus\Sigma^n_{++}:\mathcal{A}(X)=b,\ \mathcal{A}^*(y)+S=C,\ \text{for some } y\in R^m\}.$$

Central path is defined as

$$\mathcal{CP}\ :=\ \{(X,S)\in\mathcal{F}_0:\ XS=\mu I,\mu=\langle X,S\rangle/n\}.$$

Note that $(X,S)\in\mathcal{F}_0$ is on the central path iff $\left\|\frac{V^2}{\mu}-I\right\|_2=0$. Let us define the following standard neighborhoods of the central path. To make it easier for the reader to see the parallels (to the LP case), we adapted the notation and the language of LP problems and linear complementarity problems (LCP's). The notation and the language are most meaningful when one considers the eigenvalues of the underlying symmetric matrices (V^2). Let $\beta\in(0,1)$ be an absolute constant. We define
(i) so-called *wide neighborhoods*

$$\mathcal{N}^-_\infty(\beta)\ :=\ \left\{(X,S)\in\mathcal{F}_0:\ \lambda_{\min}(V^2)\geq(1-\beta)\mu\right\},$$

(ii) *infinity-norm neighborhoods*

$$\mathcal{N}_\infty(\beta)\ :=\ \left\{(X,S)\in\mathcal{F}_0:\ (1-\beta)\mu\leq\lambda_j(V^2)\leq(1+\beta)\mu,\ \forall\, j\in\{1,2,\ldots,n\}\right\},$$

(ii)$'$ or equivalently,

$$\mathcal{N}_\infty(\beta)\ :=\ \left\{(X,S)\in\mathcal{F}_0:\ \left\|\frac{V^2}{\mu}-I\right\|_2\leq\beta\right\},$$

(iii) so-called *tight (or narrow) neighborhoods*

$$\mathcal{N}_2(\beta) \;:=\; \left\{(X,S)\in\mathcal{F}_0:\; \left\|\frac{V^2}{\mu}-I\right\|_F \le \beta\right\}.$$

Note that (as is well-known),

$$\mathcal{CP}\subset\mathcal{N}_2(\beta)\subset\mathcal{N}_{-\infty}(\beta)\subset\mathcal{N}_\infty(\beta)\subset\mathcal{F}_0.$$

Next, we state a generic path-following algorithm which can utilize any of the neighborhoods above.

Generic, Strictly Monotone, Path-Following Algorithm for SDP Problems

$0 < \epsilon < 1$, $\gamma' \in (0,1)$ and $\mathcal{N} \subset \mathcal{F}_0$ (a neighborhood of the central path) must be described. $(X,S) \in \mathcal{N}$ must be given.

$k := 0$

while $\left(\langle X^{(k)}, S^{(k)}\rangle \ > \ \epsilon\langle X^{(0)}, S^{(0)}\rangle\right)$ *do*

$(X,S) := (X^{(k)}, S^{(k)})$

find $T \in \mathcal{T}(X,S)$ *corresponding to the Nesterov-Todd scaling*

compute W_p, W_q, Z_p, Z_q *and* γ_P, γ_D

$\bar{\gamma} := \gamma_P$ *if* $\gamma_P \in (0,1]$ *and* $\bar{\gamma} := \gamma_D$, *otherwise*

$\gamma := \gamma'\bar{\gamma}$

$d_x := \sqrt{\mu}T(\gamma Z_p - W_p), \quad d_s := \sqrt{\mu}T^{-1}(\gamma Z_q - W_q)$

choose maximum $\alpha \in (0,1]$ *such that* $(X + \alpha d_x, S + \alpha d_s) \in \mathcal{N}$

$\left(X^{(k+1)}, S^{(k+1)}\right) := (X + \alpha d_x, S + \alpha d_s)$

$k := k + 1$

end

□

Lemma 3.1 *Let $(X,S) \in \mathcal{N}_2(\beta)$, $\gamma' := (1 - \delta/\sqrt{n})$, and $\gamma := \gamma'\bar{\gamma}$, where $\delta > 0$ is an absolute constant. Then*

$$\|\bar{d}_x(\gamma)\bar{d}_s(\gamma)\|_F \leq \frac{\bar{\gamma}\mu(\delta+\beta)^2}{(1-\beta)}.$$

Proof. First we estimate $\|W - \gamma Z\|_2$.

$$\|W - \gamma Z\|_2 \leq (1-\gamma)\|W\|_2 + \gamma\|W - Z\|_2 \leq (1-\gamma)\sqrt{n} + \gamma\frac{\beta}{\sqrt{1-\beta}} \leq \frac{\delta+\beta}{\sqrt{1-\beta}}.$$

For the second inequality above, we used Lemma 2.3 (b). Without loss of generality, we assume $\bar{\gamma} = \gamma_P$. We have

$$\|\bar{d}_x(\gamma)\bar{d}_s(\gamma)\|_F \ = \ \mu\|(W_p - \gamma Z_p)(W_q - \gamma Z_q)\|_F$$

$$
\begin{aligned}
&\leq \mu\|W_p - \gamma'\gamma_P Z_p\|_F \|W_q - \gamma Z_q\|_2 \\
&\leq \mu\left(\|W_p - \gamma_P Z_p\|_F + \gamma_P(1-\gamma')\|Z_p\|_F\right)\frac{\delta+\beta}{\sqrt{1-\beta}} \\
&\leq \frac{\mu(\delta+\beta)}{\sqrt{1-\beta}}\left(\gamma_P\|W_p - Z_p\|_F + \gamma_P(1-\gamma')\sqrt{\frac{n}{1-\beta}}\right) \\
&\leq \frac{\mu\gamma_P(\delta+\beta)}{\sqrt{1-\beta}}\left(\frac{\beta}{\sqrt{1-\beta}} + \frac{\delta}{\sqrt{1-\beta}}\right) \\
&\leq \frac{\mu\gamma_P(\delta+\beta)^2}{1-\beta}.
\end{aligned}
$$

The second inequality uses $\|W_q - \gamma Z_q\|_F \leq \|W - \gamma Z\|_F \leq \frac{\delta+\beta}{\sqrt{1-\beta}}$. The third inequality uses the fact that $\|Z_p\|_F \leq \|Z\|_F \leq \sqrt{n}\|Z\|_2$. Then by Definition 2.1 (d) and the fact that $(X,S) \in \mathcal{N}_2(\beta)$ we have $\|Z\|_2 \leq 1/\sqrt{1-\beta}$. The fourth inequality follows from Lemma 2.3 (b) and $(X,S) \in \mathcal{N}_2(\beta)$. □

Corollary 3.1 *Let $(X,S) \in \mathcal{N}_2(\beta)$, $\gamma' := (1-\delta/\sqrt{n})$, and $\gamma := \gamma'\bar{\gamma}$, where $\delta \geq 1$ is an absolute constant. Then $(X + \alpha d_x(\gamma), S + \alpha d_s(\gamma)) \in \mathcal{N}_2(\beta)$, for all $\alpha \in \left[0, \frac{\gamma'\beta(1-\beta)}{(\delta+\beta)^2}\right]$.*

Proof.

$$
\begin{aligned}
\left\|\frac{(V+\alpha\bar{d}_x)(V+\alpha\bar{d}_s)}{\mu} - [1-\alpha(1-\gamma)]\,I\right\|_F &\leq (1-\alpha)\beta + \frac{\alpha^2}{\mu}\|\bar{d}_x(\gamma)\bar{d}_s(\gamma)\|_F \\
&\leq (1-\alpha)\beta + \alpha^2\frac{\bar{\gamma}(\delta+\beta)^2}{1-\beta} \\
&\leq (1-\alpha)\beta + \alpha\beta\gamma \\
&= [1-\alpha(1-\gamma)]\,\beta
\end{aligned}
$$

The first inequality follows from Lemma 2.4 and the fact that $\alpha \leq 1$. The second inequality follows from Lemma 3.1. Lemma 2.1 of Monteiro [19] yields,

$$
\left\|\frac{T(V+\alpha\bar{d}_x)T^{-1}(V+\alpha\bar{d}_s)}{\mu} - [1-\alpha(1-\gamma)]\,I\right\|_F \leq \left\|\frac{(V+\alpha\bar{d}_x)(V+\alpha\bar{d}_s)}{\mu} - [1-\alpha(1-\gamma)]\,I\right\|_F
$$

from which we conclude $(X + \alpha d_x(\gamma), S + \alpha d_s(\gamma)) \in \mathcal{N}_2(\beta)$, for all $\alpha \in \left[0, \frac{\gamma'\beta(1-\beta)}{(\delta+\beta)^2}\right]$. (The fact that $(X + \alpha d_x(\gamma))$ and $(S + \alpha d_s(\gamma))$ are positive definite follows from a continuity argument and the neighborhood condition which we verified.) □

Lemma 3.2 *If* $(X,S) \in \mathcal{N}_\infty^-(\beta)$ *then*

$$\|W - \gamma Z\|_F \;\leq\; \sqrt{\frac{2n}{1-\beta}}, \quad \textit{for all} \quad \gamma \in [0,1].$$

Moreover, if $\bar{\gamma} = \gamma_P$ *then*

$$\|W_p - \gamma' \bar{\gamma} Z_p\|_F \;\leq\; (1+\sqrt{2})\bar{\gamma}\sqrt{\frac{n}{1-\beta}}, \quad \textit{for all} \quad \gamma' \in (0,1].$$

Proof.

$$\|W - \gamma Z\|_F^2 \;=\; n - 2\gamma n + \gamma^2 \langle Z, Z\rangle \;\leq\; n + \gamma^2 \frac{n}{1-\beta} \;\leq\; \frac{2n}{1-\beta}.$$

For the second part of the statement of the lemma, we have

$$\begin{aligned}
\|W - \bar{\gamma} Z\|_F &\leq \|W_p - \gamma_P Z_p\|_F + \gamma_P(1-\gamma')\|Z_p\|_F \\
&\leq \gamma_P \left(\|W_p - Z_p\|_F + \|Z_p\|_F\right) \\
&\leq \gamma_P \left(\|W - Z\|_F + \|Z\|_F\right) \\
&\leq \gamma_P \left(\sqrt{\frac{2n}{1-\beta}} + \sqrt{\frac{n}{1-\beta}}\right) \\
&= \gamma_P (1+\sqrt{2})\sqrt{\frac{n}{1-\beta}}.
\end{aligned}$$

The second inequality uses Lemma 2.2. The fourth inequality uses Lemma 2.3 (b), Definition 2.1 (d) and the fact that $(X,S) \in \mathcal{N}_\infty^-(\beta)$. □

Lemma 3.3 *Let* $\gamma' \in (0,1]$ *be an absolute constant. If* $(X,S) \in \mathcal{N}_\infty^-(\beta)$, $\gamma := \gamma'\bar{\gamma}$, *and* $\alpha \in \left[0, \frac{\gamma'\beta(1-\beta)}{4n}\right]$ *then* $(X + \alpha d_x(\gamma), S + \alpha d_s(\gamma)) \in \mathcal{N}_\infty^-(\beta)$.

Proof. We use the relation (2), to obtain the following inequality

$$\begin{aligned}
\lambda_{\min}\left(\frac{(V+\alpha\bar{d}_x)(V+\alpha\bar{d}_s)}{\mu}\right) &\geq \lambda_{\min}\left((1-\alpha)\frac{V^2}{\mu} + \alpha\gamma I\right) + \frac{\alpha^2}{2\mu}\lambda_{\max}\left(\bar{d}_x\bar{d}_s + \bar{d}_s\bar{d}_x\right) \\
&\geq (1-\alpha)(1-\beta) + \alpha\gamma - \alpha^2\|W_p - \gamma Z_p\|_F\|W_q - \gamma Z_q\|_F \\
&\geq (1-\alpha)(1-\beta) + \alpha\gamma - \alpha^2\frac{4\bar{\gamma}n}{1-\beta} \\
&\geq (1-\alpha)(1-\beta) + \alpha\gamma - \alpha\gamma'\bar{\gamma}\beta \\
&= \left[1 - \alpha(1-\gamma)\right](1-\beta).
\end{aligned}$$

To obtain the third inequality above, we used Lemma 3.2. To conclude that $(X + \alpha d_x(\gamma), S + \alpha d_s(\gamma)) \in \mathcal{N}_\infty^-(\beta)$, it suffices to establish

$$\lambda_{\min}\left(\frac{(X + \alpha d_x(\gamma))(S + \alpha d_s(\gamma))}{\mu}\right) \;\geq\; \lambda_{\min}\left(\frac{(V + \alpha \bar{d}_x)(V + \alpha \bar{d}_s)}{\mu}\right)$$

which follows from Lemma 4.4 of Monteiro and Zhang [21]. (The fact that $(X + \alpha d_x(\gamma))$ and $(S + \alpha d_s(\gamma))$ are positive definite follows from a continuity argument and the neighborhood condition which we verified.) □

Lemma 3.4 *Let $\gamma' \in (0, 1]$ be an absolute constant. If $(X, S) \in \mathcal{N}_\infty(\beta)$, $\gamma := \gamma'\bar{\gamma}$, and $\alpha \in \left[0, \frac{\gamma'\beta(1-\beta)}{4n}\right]$ then $(X + \alpha d_x(\gamma), S + \alpha d_s(\gamma)) \in \mathcal{N}_\infty(\beta)$.*

Proof. By Lemma 2.3 (b) and Lemma 3.2, we have

$$\begin{aligned}\left\|\frac{(V + \alpha \bar{d}_x)(V + \alpha \bar{d}_s)}{\mu} - [1 - \alpha(1-\gamma)]\, I\right\|_2 &\leq (1-\alpha)\left\|\frac{V^2}{\mu} - I\right\|_2 + \alpha^2 \frac{4\bar{\gamma}n}{1-\beta} \\ &\leq (1-\alpha)\beta + \alpha\beta\gamma \\ &= [1 - \alpha(1-\gamma)]\,\beta\end{aligned}$$

To conclude that $(X + \alpha d_x(\gamma), S + \alpha d_s(\gamma)) \in \mathcal{N}_\infty(\beta)$, it suffices to establish

$$\left\|\frac{(X + \alpha d_x(\gamma))(S + \alpha d_s(\gamma))}{\mu} - [1 - \alpha(1-\gamma)]\, I\right\|_2 \;\leq\; \left\|\frac{(V + \alpha \bar{d}_x)(V + \alpha \bar{d}_s)}{\mu} - [1 - \alpha(1-\gamma)]\, I\right\|_2$$

which follows from Lemma 4.4 of Monteiro and Zhang [21]. (The fact that $(X + \alpha d_x(\gamma))$ and $(S + \alpha d_s(\gamma))$ are positive definite follows from a continuity argument and the neighborhood condition which we verified.) □

Theorem 3.1 *Let $\beta, \gamma' \in (0, 1)$ and $\delta \geq 1$ be absolute constants. Then the primal-dual path-following algorithms based on Nesterov-Todd search directions and the neighborhoods $\mathcal{N}_\infty^-(\beta)$, $\mathcal{N}_\infty(\beta)$ and $\mathcal{N}_2(\beta)$ achieve $O(n\ln(1/\epsilon))$, $O(n\ln(1/\epsilon))$ and $O(\sqrt{n}\ln(1/\epsilon))$ iteration bounds and are strictly monotone in both primal and dual objectives if the centering parameter γ is chosen as $\gamma'\bar{\gamma}$, $\gamma'\bar{\gamma}$ and $(1 - \delta/\sqrt{n})\bar{\gamma}$ respectively.*

Proof. First, consider the algorithms using the wider neighborhoods $\mathcal{N}_\infty(\beta)$ and $\mathcal{N}_\infty^-(\beta)$. For both of these neighborhoods the rest of our analysis is the same. We illustrate it on $\mathcal{N}_\infty^-(\beta)$ neighborhood. We use Lemmas 2.3 and 3.3.

$$\frac{\langle X^{(k+1)}, S^{(k+1)}\rangle}{\langle X^{(k)}, S^{(k)}\rangle} \;\leq\; 1 - \frac{\beta\gamma'(1-\gamma)(1-\beta)}{4n} \;=\; 1 - \Omega(1/n).$$

Therefore, in $O(n\ln(1/\epsilon))$ iterations we must have

$$\frac{\langle X^{(k+1)}, S^{(k+1)}\rangle}{\langle X^{(0)}, S^{(0)}\rangle} \;\leq\; \epsilon.$$

Similarly, for the path following algorithm using the tight neighborhood, $\mathcal{N}_2(\beta)$ and, utilizing Lemma 2.3 and Corollary 3.1, we find

$$\frac{\langle X^{(k+1)}, S^{(k+1)}\rangle}{\langle X^{(k)}, S^{(k)}\rangle} \;\leq\; 1 - \frac{\beta\gamma'(1-\gamma)(1-\beta)}{(\delta+\beta)^2} \;\leq\; 1 - \frac{\beta\gamma'(1-\gamma')(1-\beta)}{(\delta+\beta)^2} \;=\; 1 - \Omega(1/\sqrt{n}).$$

Therefore, in $O(\sqrt{n}\ln(1/\epsilon))$ iterations we must have

$$\frac{\langle X^{(k+1)}, S^{(k+1)}\rangle}{\langle X^{(0)}, S^{(0)}\rangle} \;\leq\; \epsilon.$$

Strict monotonicity of all these algorithms follows from Theorem 2.2. □

4 Concluding Remarks

We presented strictly monotone primal-dual interior-point algorithms which achieve the current best iteration bounds. Our algorithms compute two components (affine-scaling and centering) of the search directions separately. Then, based on these components, the algorithms compute the critical values primal and dual centering coefficients (γ_P and γ_D). Based on those values, the algorithms take a proper combination of the two components as the search direction. Even though we solve two linear systems of equations, in each iteration, the coefficient matrix of the linear systems is the same. As a result, the amount of extra computational work required by our algorithm is of lower order than that required to solve a single linear system and therefore is negligible. For details, see [27] and [8].

In a practical version of the algorithm, we would allow much more aggressive choices in the selection of the step size α. In a practical version, we may also allow different step sizes in the primal and dual spaces. If we do not confine the iterates of the algorithm to a neighborhood of the central path, then our complexity analyses here, do not directly apply. However, the search directions still keep the desired properties discussed in the introduction.

In order to apply our method for strict monotonicity, we need the current point to be feasible. It is true that in many semidefinite programming problems no interior-feasible solution is available for an initial point. Even in such a case, we often get primal and/or dual feasible solutions in earlier iterations, so that we can then apply our method. Also there are many important problems in which initial interior-feasible solutions are available; e.g., norm minimization problems, Chebyshev problems, maximum cut problems and graph partitioning problems.

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