A Note on the Nesterov-Todd and the Kojima-Shindoh-Hara Search Directions in Semidefinite Programming

Masakazu Kojima[†]
Masayuki Shida[‡]
Susumu Shindoh[♯]

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Abstract. This short note shows that the Nesterov-Todd search direction used in primal-dual interior-point methods for semidefinite programs belongs to the family of search directions proposed by Kojima, Shindoh and Hara.

Key words Semidefinite Programming, Interior-Point Method, Search Direction

† Department of Mathematical and Computing Sciences
Tokyo Institute of Technology,
2-12-1 Oh-Okayama, Meguro-ku, Tokyo 152, Japan.
e-mail: kojima@is.titech.ac.jp

‡ Department of Mathematics, Faculty of Engineering, Kanagawa University,
Rokkakubashi, Kanagawa-ku, Yokohama 221, Japan.
e-mail: shida@is.titech.ac.jp

♯ Department of Mathematics and Physics, The National Defense Academy,
Hashirimizu 1-10-20, Yokosuka, 239, Japan.
e-mail: shindoh@cc.nda.ac.jp

1 Introduction.

Recently there have been proposed many primal-dual interior-point methods for semidefinite programs. Among various search directions used in those methods, we are concerned with the NT (Nesterov-Todd) search direction [6, 7] and the KSH (Kojima-Shindoh-Hara) family of search directions [4]. The purpose of this short note is to show that the NT search direction belongs to the KSH family. Our proof of this fact is based on the symmetric primal-dual transformation (or scaling) [9] and the scale invariance [10] of these search directions.

We use $\mathcal{S}$ for the set of all $n \times n$ symmetric real matrices, and $\tilde{\mathcal{S}}$ for the set of all $n \times n$ skew-symmetric matrices. We write $\boldsymbol{X} \succ \boldsymbol{O}$ if $\boldsymbol{X} \in \mathcal{S}$ is positive definite, and $\boldsymbol{X} \succeq \boldsymbol{O}$ if $\boldsymbol{X} \in \mathcal{S}$ is positive semidefinite. Here $\boldsymbol{O}$ denotes the $n \times n$ zero matrix. The notation $\boldsymbol{A} \bullet \boldsymbol{B}$ stands for the inner product of two $n \times n$ real matrix, *i.e.*, $\boldsymbol{A} \bullet \boldsymbol{B} = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} B_{ij}$. We consider the standard form SDP (semidefinite program) and its dual:

$$
\begin{array}{lll}
\text{(SDP)} & \text{minimize} & \boldsymbol{C} \bullet \boldsymbol{X} \\
 & \text{subject to} & \boldsymbol{A}_i \bullet \boldsymbol{X} = b_i \ (i = 1, 2, \ldots, m) \ \text{ and } \boldsymbol{X} \succeq \boldsymbol{O}. \\
\text{(SDD)} & \text{maximize} & \sum_{i=1}^{m} b_i y_i \\
 & \text{subject to} & \sum_{i=1}^{m} \boldsymbol{A}_i y_i + \boldsymbol{Y} = \boldsymbol{C} \ \text{ and } \boldsymbol{Y} \succeq \boldsymbol{O}.
\end{array}
$$

Here $\boldsymbol{A}_i \in \mathcal{S}\ (i = 1, 2, \ldots, m)$ and $\boldsymbol{C} \in \mathcal{S}$ denote constant matrices, and $\boldsymbol{b} = (b_1, b_2, \ldots, b_m)^T$ a constant vector in the m-dimensional Euclidean space $\mathbb{R}^m$.

2 Main Result.

Let $\boldsymbol{X} \succ \boldsymbol{O}$, $\boldsymbol{Y} \succ \boldsymbol{O}$ and $\boldsymbol{y} \in \mathbb{R}^m$ be fixed throughout this section. Let $\mu = \boldsymbol{X} \bullet \boldsymbol{Y}/n$ and $\beta \in [0,\ 1]$.

The NT search direction [6, 7] at $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{y})$ is given as a solution $(\Delta\boldsymbol{X}, \Delta\boldsymbol{Y}, \Delta\boldsymbol{y})$ to the system of equations

$$
\boldsymbol{A}_i \bullet \Delta\boldsymbol{X} = b_i - \boldsymbol{A}_i \bullet \boldsymbol{X} \ (i = 1, 2, \ldots . m), \tag{1}
$$

$$
\sum_{i=1}^{m} \boldsymbol{A}_i \Delta y_i + \Delta\boldsymbol{Y} = \boldsymbol{C} - \sum_{i=1}^{m} \boldsymbol{A}_i y_i - \boldsymbol{Y}, \tag{2}
$$

$$
\boldsymbol{W}^{-1} \Delta\boldsymbol{X}\, \boldsymbol{W}^{-1} + \Delta\boldsymbol{Y} = \beta\mu \boldsymbol{X}^{-1} - \boldsymbol{Y}. \tag{3}
$$

Here $\boldsymbol{W} = \boldsymbol{X}^{1/2}(\boldsymbol{X}^{1/2}\boldsymbol{Y}\boldsymbol{X}^{1/2})^{-1/2}\boldsymbol{X}^{1/2} = \boldsymbol{Y}^{-1/2}(\boldsymbol{Y}^{1/2}\boldsymbol{X}\boldsymbol{Y}^{1/2})^{1/2}\boldsymbol{Y}^{-1/2} \in \mathcal{S}$.

On the other hand, the KSH family of search directions [4] at $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{y})$ is given as the set of solutions $(\Delta\boldsymbol{X}, \Delta\boldsymbol{Y}, \Delta\boldsymbol{y})$ to the system of equations (1), (2) and

$$
\begin{aligned}
&(\Delta\boldsymbol{X} + \widetilde{\Delta\boldsymbol{X}})\boldsymbol{Y} + \boldsymbol{X}(\Delta\boldsymbol{Y} + \widetilde{\Delta\boldsymbol{Y}}) = \beta\mu\boldsymbol{I} - \boldsymbol{X}\boldsymbol{Y} \\
&\text{for some } (\widetilde{\Delta\boldsymbol{X}}, \widetilde{\Delta\boldsymbol{Y}}) \in \tilde{\mathcal{S}} \times \tilde{\mathcal{S}} \text{ satisfying } \widetilde{\Delta\boldsymbol{X}} \bullet \widetilde{\Delta\boldsymbol{Y}} \geq 0.
\end{aligned} \tag{4}
$$

We call a linear subspace $\tilde{\mathcal{A}}$ of $\tilde{\mathcal{S}} \times \tilde{\mathcal{S}}$ *monotone* if $\widetilde{\Delta\boldsymbol{X}} \bullet \widetilde{\Delta\boldsymbol{Y}} \geq 0$ for every pair $(\widetilde{\Delta\boldsymbol{X}}, \widetilde{\Delta\boldsymbol{Y}}) \in \tilde{\mathcal{A}}$. If we assume $\widetilde{\Delta\boldsymbol{X}} \bullet \widetilde{\Delta\boldsymbol{Y}} \geq 0$ for some $(\widetilde{\Delta\boldsymbol{X}}, \widetilde{\Delta\boldsymbol{Y}}) \in \tilde{\mathcal{S}} \times \tilde{\mathcal{S}}$ as in (4), $\tilde{\mathcal{A}} = \{\alpha(\widetilde{\Delta\boldsymbol{X}}, \widetilde{\Delta\boldsymbol{Y}}) : \alpha \in \mathbb{R}\}$ forms a 0 or 1 dimensional monotone linear subspace of $\tilde{\mathcal{S}} \times \tilde{\mathcal{S}}$, and we can extend $\tilde{\mathcal{A}}$ to some maximal (*i.e.*, $n(n-1)/2$ dimensional) monotone linear subspace $\tilde{\mathcal{F}}$ of $\tilde{\mathcal{S}} \times \tilde{\mathcal{S}}$. See, for example, Section 2 of [8]. Therefore $(\Delta\boldsymbol{X}, \Delta\boldsymbol{Y})$ satisfies (4) if and only if there exists a

maximal monotone linear subspace $\widetilde{\mathcal{F}}$ of $\widetilde{\mathcal{S}} \times \widetilde{\mathcal{S}}$ such that

$$(\Delta \boldsymbol{X} + \widetilde{\Delta \boldsymbol{X}})\boldsymbol{Y} + \boldsymbol{X}(\Delta \boldsymbol{Y} + \widetilde{\Delta \boldsymbol{Y}}) = \beta\mu \boldsymbol{I} - \boldsymbol{X}\boldsymbol{Y} \quad \text{for some } (\widetilde{\Delta \boldsymbol{X}}, \widetilde{\Delta \boldsymbol{Y}}) \in \widetilde{\mathcal{F}}. \tag{4}'$$

In fact, we have already seen the "only if" part above, and "if" part is straightforward.

As a special case, we can choose $\widetilde{\mathcal{F}} = \{0\} \times \widetilde{\mathcal{S}}$, and then (4)' becomes

$$\Delta \boldsymbol{X}\, \boldsymbol{Y} + \boldsymbol{X}(\Delta \boldsymbol{Y} + \widetilde{\Delta \boldsymbol{Y}}) = \beta\mu \boldsymbol{I} - \boldsymbol{X}\boldsymbol{Y} \quad \text{for some } \widetilde{\Delta \boldsymbol{Y}} \in \widetilde{\mathcal{S}}. \tag{5}$$

It is known that the solution to the system of equations (1), (2) and (5) coincides with the search direction proposed by Monteiro [5]. See also Zhang [11]. The lemma below shows that the NT search direction $(\Delta \boldsymbol{X}, \Delta \boldsymbol{Y}, \Delta \boldsymbol{y})$ also belongs to the KSH family of search directions.

Lemma. *Let $\boldsymbol{L}$ be an $n \times n$ nonsingular matrix such that $\boldsymbol{W} = \boldsymbol{L}\boldsymbol{L}^T$ and*

$$\widetilde{\mathcal{F}} = \{(\boldsymbol{L}\widetilde{\boldsymbol{U}}\boldsymbol{L}^T, \boldsymbol{L}^{-T}\widetilde{\boldsymbol{U}}\boldsymbol{L}^{-1}) \ : \ \widetilde{\boldsymbol{U}} \in \widetilde{\mathcal{S}}\}. \tag{6}$$

Then the following (i) and (ii) hold.

(i) $\widetilde{\mathcal{F}}$ is a maximal monotone linear subspace of $\widetilde{\mathcal{S}} \times \widetilde{\mathcal{S}}$.

(ii) $(\Delta \boldsymbol{X}, \Delta \boldsymbol{Y})$ satisfies (3) if and only if it satisfies (4)' with $\widetilde{\mathcal{F}}$ defined in (6).

Proof: The assertion (i) is straightforward, so that we only prove (ii). Applying the symmetric primal-dual transformation [9] (see also [6])

$$(\boldsymbol{X}, \boldsymbol{Y}) \mapsto (\boldsymbol{V}, \boldsymbol{V}) = (\boldsymbol{L}^{-1}\boldsymbol{X}\boldsymbol{L}^{-T}, \boldsymbol{L}^T\boldsymbol{Y}\boldsymbol{L}) \tag{7}$$

to equation (3), we have

$$d\boldsymbol{X} + d\boldsymbol{Y} = \beta\mu \boldsymbol{V}^{-1} - \boldsymbol{V},$$

where $(d\boldsymbol{X}, d\boldsymbol{Y}) = (\boldsymbol{L}^{-1}\Delta \boldsymbol{X}\, \boldsymbol{L}^{-T}, \boldsymbol{L}^T\Delta \boldsymbol{Y}\, \boldsymbol{L})$. Hence, if $(\Delta \boldsymbol{X}, \Delta \boldsymbol{Y})$ is a solution of (3), then $(d\boldsymbol{X}, d\boldsymbol{Y})$ is a solution of

$$(d\boldsymbol{X}\, \boldsymbol{V} + \boldsymbol{V}\, d\boldsymbol{Y}) + (d\boldsymbol{X}\, \boldsymbol{V} + \boldsymbol{V}\, d\boldsymbol{Y})^T = 2(\beta\mu \boldsymbol{I} - \boldsymbol{V}^2). \tag{3}'$$

Sturm and Zhang (see Section 3 of [9]) showed that the converse is also true, and so $(\Delta \boldsymbol{X}, \Delta \boldsymbol{Y})$ is a solution of (3) if and only if $(d\boldsymbol{X}, d\boldsymbol{Y})$ is a solution of (3)'. Applying the same symmetric primal-dual transformation (7) to (4)' with $\widetilde{\mathcal{F}}$ defined in (6), we have

$$(d\boldsymbol{X} + \widetilde{\boldsymbol{U}})\boldsymbol{V} + \boldsymbol{V}(d\boldsymbol{Y} + \widetilde{\boldsymbol{U}}) = \beta\mu \boldsymbol{I} - \boldsymbol{V}^2 \quad \text{for some } \widetilde{\boldsymbol{U}} \in \widetilde{\mathcal{S}}. \tag{4}''$$

Here $(\Delta \boldsymbol{X}, \Delta \boldsymbol{Y})$ is a solution of (4)' if and only if $(d\boldsymbol{X}, d\boldsymbol{Y}) = (\boldsymbol{L}^{-1}\Delta \boldsymbol{X}\, \boldsymbol{L}^T, \boldsymbol{L}^T\Delta \boldsymbol{Y}\, \boldsymbol{L})$ is a solution of (4)". Therefore it suffices to show that $(d\boldsymbol{X}, d\boldsymbol{Y})$ is a solution of (3)' if and only if it is a solution of (4)". We first show the "if" part. Suppose that $(d\boldsymbol{X}, d\boldsymbol{Y})$ is a solution of (4)". Then

$$\begin{aligned}
& 2\left(\beta\mu \boldsymbol{I} - \boldsymbol{V}^2\right) \\
&= \left((d\boldsymbol{X} + \widetilde{\boldsymbol{U}})\boldsymbol{V} + \boldsymbol{V}(d\boldsymbol{Y} + \widetilde{\boldsymbol{U}})\right) + \left((d\boldsymbol{X} + \widetilde{\boldsymbol{U}})\boldsymbol{V} + \boldsymbol{V}(d\boldsymbol{Y} + \widetilde{\boldsymbol{U}})\right)^T \\
&= (d\boldsymbol{X}\, \boldsymbol{V} + \boldsymbol{V}\, d\boldsymbol{Y}) + (d\boldsymbol{X}\, \boldsymbol{V} + \boldsymbol{V}\, d\boldsymbol{Y})^T + \left(\widetilde{\boldsymbol{U}}\boldsymbol{V} + \boldsymbol{V}\widetilde{\boldsymbol{U}}\right) + \left(\widetilde{\boldsymbol{U}}\boldsymbol{V} + \boldsymbol{V}\widetilde{\boldsymbol{U}}\right)^T \\
&= (d\boldsymbol{X}\, \boldsymbol{V} + \boldsymbol{V}\, d\boldsymbol{Y}) + (d\boldsymbol{X}\, \boldsymbol{V} + \boldsymbol{V}\, d\boldsymbol{Y})^T + \left(\widetilde{\boldsymbol{U}}\boldsymbol{V} + \boldsymbol{V}\widetilde{\boldsymbol{U}}\right) + \left(-\boldsymbol{V}\widetilde{\boldsymbol{U}} - \widetilde{\boldsymbol{U}}\boldsymbol{V}\right) \\
&\quad \text{(since } \widetilde{\boldsymbol{U}} \in \widetilde{\mathcal{S}} \text{ and } \boldsymbol{V} \in \mathcal{S}) \\
&= (d\boldsymbol{X}\, \boldsymbol{V} + \boldsymbol{V}\, d\boldsymbol{Y}) + (d\boldsymbol{X}\, \boldsymbol{V} + \boldsymbol{V}\, d\boldsymbol{Y})^T.
\end{aligned}$$

Thus we have shown that $(\boldsymbol{dX}, \boldsymbol{dY})$ satisfies (3)'.

To prove the "only if" part, assume that $(\boldsymbol{dX}, \boldsymbol{dY}) \in \mathcal{S} \times \mathcal{S}$ satisfies (3)'. Consider a Lyapunov type equation

$$2(\widetilde{\boldsymbol{U}}\boldsymbol{V} + \boldsymbol{V}\widetilde{\boldsymbol{U}}) = -(\boldsymbol{dX}\,\boldsymbol{V} + \boldsymbol{V}\,\boldsymbol{dY}) + (\boldsymbol{dX}\,\boldsymbol{V} + \boldsymbol{V}\,\boldsymbol{dY})^T \tag{8}$$

in an $n \times n$ matrix $\widetilde{\boldsymbol{U}}$. Note that the coefficient matrix $\boldsymbol{V} = \boldsymbol{L}^{-1}\boldsymbol{X}\boldsymbol{L}^{-T} = \boldsymbol{L}^T\boldsymbol{Y}\boldsymbol{L} \in \mathcal{S}$ is positive definite and the right hand side matrix is skew symmetric. Using the Kronecker product (see, for example, [3]), we can prove that equation (8) has a unique solution $\widetilde{\boldsymbol{U}} \in \widetilde{\mathcal{S}}$. Now we see that

$$\begin{aligned} 2(\beta\mu\boldsymbol{I} - \boldsymbol{V}^2) &= 2(\boldsymbol{dX}\,\boldsymbol{V} + \boldsymbol{V}\,\boldsymbol{dY}) - (\boldsymbol{dX}\,\boldsymbol{V} + \boldsymbol{V}\,\boldsymbol{dY}) + (\boldsymbol{dX}\,\boldsymbol{V} + \boldsymbol{V}\,\boldsymbol{dY})^T \\ &= 2(\boldsymbol{dX}\,\boldsymbol{V} + \boldsymbol{V}\,\boldsymbol{dY}) + 2(\widetilde{\boldsymbol{U}}\boldsymbol{V} + \boldsymbol{V}\widetilde{\boldsymbol{U}}) \quad \text{(by (8))} \\ &= 2\left((\boldsymbol{dX} + \widetilde{\boldsymbol{U}})\boldsymbol{V} + \boldsymbol{V}(\boldsymbol{dY} + \widetilde{\boldsymbol{U}})\right). \end{aligned}$$

Thus we have shown that $(\boldsymbol{dX}, \boldsymbol{dY})$ satisfies (4)". ∎

3 Concluding Remark.

Kojima, Shindoh and Hara [4] presented an infeasible-interior-point potential reduction method for the monotone semidefinite linear complementarity problem which includes a primal-dual pair of SDPs as a special case. The method can employ any search direction in the KSH family, hence, the NT search direction, at each iteration. Therefore its global convergence and its complexity analysis remain valid for the NT search direction.

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