Monotone Semidefinite Complementarity Problems

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Abstract. In this paper, we study some basic properties of the monotone semidefinite nonlinear complementarity problem (SDCP). We show that the trajectory continuously accumulates into the solution set of the SDCP passing through the set of the infeasible but positive definite matrices under certain conditions. Especially, for the monotone semidefinite linear complementarity problem, the trajectory converges to an analytic center of the solution set, provided that there exists a strictly complementary solution. Finally, we propose the globally convergent infeasible-interior-point algorithm for the SDCP.

Key words Monotone Semidefinite Complementarity Problem, Trajectory, Interior Point Algorithm

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1 Introduction.

Let $\mathcal{M}(n)$ and $\mathcal{S}(n)$ denote the class of $n \times n$ real matrices and the class of $n \times n$ symmetric real matrices, respectively. Assume that $\boldsymbol{A},\ \boldsymbol{B} \in \mathcal{M}(n)$. The notation $\boldsymbol{A} \succeq \boldsymbol{B}$ (respectively, $\boldsymbol{A} \succ \boldsymbol{B}$) means that $\boldsymbol{A}-\boldsymbol{B}$ is positive semidefinite, *i.e.*, $\boldsymbol{x}^\top(\boldsymbol{A}-\boldsymbol{B})\boldsymbol{x} \geq 0$ for every $\boldsymbol{x} \in \mathcal{R}^n$ (respectively, positive definite, *i.e.*, $\boldsymbol{x}^\top(\boldsymbol{A}-\boldsymbol{B})\boldsymbol{x} > 0$ for every nonzero $\boldsymbol{x} \in \mathcal{R}^n$), where $\mathcal{R}^n$ is the n-dimensional Euclidean space. We use $\boldsymbol{X} \bullet \boldsymbol{Y} := \mathbf{Tr}\ \boldsymbol{X}^\top\boldsymbol{Y}$ (trace of $\boldsymbol{X}^\top\boldsymbol{Y}$) as the inner product of $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $\mathcal{M}(n)$. For two matrices $\boldsymbol{A}^1,\ \boldsymbol{A}^2$, define $\mathbf{diag}\{\boldsymbol{A}^1, \boldsymbol{A}^2\} := \begin{pmatrix} \boldsymbol{A}^1 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{A}^2 \end{pmatrix}$, where $\boldsymbol{O}$ is the zero matrix of an appropriate size. Inductively, for l matrices $\boldsymbol{A}^1, \boldsymbol{A}^2, \cdots, \boldsymbol{A}^l$, define $\mathbf{diag}\{\boldsymbol{A}^1, \boldsymbol{A}^2, \cdots, \boldsymbol{A}^l\}$ by

$$\mathbf{diag}\{\boldsymbol{A}^1, \boldsymbol{A}^2, \cdots, \boldsymbol{A}^l\} := \mathbf{diag}\{\boldsymbol{A}^1, \mathbf{diag}\{\boldsymbol{A}^2, \cdots, \boldsymbol{A}^l\}\}.$$

Let $\mathcal{M}(n^1, \cdots, n^l)$ and $\mathcal{S}(n^1, \cdots, n^l)$ be the class of the block-diagonal matrices and the class of the block-diagonal symmetric matrices, respectively, *i.e.*,

$$\begin{aligned} \mathcal{M}(n^1, \cdots, n^l) &:= \mathbf{diag}\{\mathcal{M}(n^1), \cdots, \mathcal{M}(n^l)\}, \\ \mathcal{S}(n^1, \cdots, n^l) &:= \mathbf{diag}\{\mathcal{S}(n^1), \cdots, \mathcal{S}(n^l)\}. \end{aligned}$$

For simplicity, we abbreviate $\mathcal{S}(n^1, \cdots, n^l)$ to $\mathcal{S}$.

This paper deals with a monotone semidefinite complementarity problem (SDCP) in $\mathcal{S}$ (or a monotone complementarity problem over the positive semidefinite cone in $\mathcal{S}$):

$$\begin{array}{lll} \textbf{SDCP} & \text{Find an} & (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_+ \times \mathcal{S}_+, \\ & \text{such that} & (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F} \ \text{ and } \ \boldsymbol{X} \bullet \boldsymbol{Y} = 0, \end{array} \tag{1}$$

where $\mathcal{S}_+ := \{\boldsymbol{X} \in \mathcal{S} : \boldsymbol{X} \succeq \boldsymbol{O}\}$ and $\mathcal{F}$ is a *maximal monotone* subset of $\mathcal{S} \times \mathcal{S}$, *i.e.*, $(\boldsymbol{X} - \boldsymbol{X}') \bullet (\boldsymbol{Y} - \boldsymbol{Y}') \geq 0$ for every $(\boldsymbol{X}, \boldsymbol{Y}),\ (\boldsymbol{X}', \boldsymbol{Y}') \in \mathcal{F}$ (monotonicity) and there is no monotone set which properly contains $\mathcal{F}$ (maximality). The SDCP (1) is reformulated as;

$$\begin{array}{lll} \textbf{SDCP} & \text{Find an} & (\boldsymbol{X}^q, \boldsymbol{Y}^q) \in \mathcal{S}(n^q)_+ \times \mathcal{S}(n^q)_+ \quad \text{for } q \in \{1, \cdots, l\} \\ & \text{such that} & (\boldsymbol{X}, \boldsymbol{Y}) = (\mathbf{diag}\{\boldsymbol{X}^1, \cdots, \boldsymbol{X}^l\}, \mathbf{diag}\{\boldsymbol{Y}^1, \cdots, \boldsymbol{Y}^l\}) \in \mathcal{F}, \\ & & \boldsymbol{X}^q \bullet \boldsymbol{Y}^q = 0 \quad \text{for } q \in \{1, \cdots, l\} \end{array}$$

where $\mathcal{S}(n^q)_+ := \{\boldsymbol{X}^q \in \mathcal{S}(n^q) : \boldsymbol{X}^q \succeq \boldsymbol{O}\}$. In the SDCP (1), it is convenient to use the class $\mathcal{S}$ of the block-diagonal symmetric matrices as the defining space instead of the class $\mathcal{S}(n)$ of all $n \times n$-symmetric matrices as the following example shows. Consider the following convex program over the positive semidefinite cone in $\mathcal{S}(m)$:

$$\begin{array}{ll} \text{minimize} & f_0(\boldsymbol{X}) \\ \text{subject to} & f_i(\boldsymbol{X}) \leq 0\ (1 \leq i \leq r),\ f_j(\boldsymbol{X}) = 0\ (r+1 \leq j \leq s),\ \boldsymbol{X} \in \mathcal{S}(m)_+, \end{array} \tag{2}$$

where $f_i(\boldsymbol{X})$ $(i = 0, \cdots, r)$ are differentiable convex functions and $f_j(\boldsymbol{X})$ $(j = r+1, \cdots, s)$ are affine functions. The Lagrange function $\mathcal{L}$ of the convex program (2) is

$$\mathcal{L} = f_0(\boldsymbol{X}) + \sum \mu_i f_i(\boldsymbol{X}) + \sum \nu_j f_j(\boldsymbol{X}) - \boldsymbol{X} \bullet \boldsymbol{Y}$$

with multipliers $\mu_i \in \mathcal{R}_+ := \{\mu \in \mathcal{R} : \mu \geq 0\}$, $\nu_j \in \mathcal{R}$ and $\boldsymbol{Y} \in \mathcal{S}(m)_+$. Under a certain constraint qualification, *e.g.*, the Slater condition [34], $\boldsymbol{X}$ is optimal if and only if the Karush-Kuhn-Tucker (KKT) condition for the convex program (2) holds at $\boldsymbol{X}$, *i.e.*,

$$\begin{array}{l} D\mathcal{L} = Df_0(\boldsymbol{X}) + \sum \mu_i Df_i(\boldsymbol{X}) + \sum \nu_j Df_j(\boldsymbol{X}) - \boldsymbol{Y} = \boldsymbol{O}, \\ \mu_i \geq 0,\ f_i(\boldsymbol{X}) \leq 0,\ \mu_i f_i(\boldsymbol{X}) = 0,\ f_j(\boldsymbol{X}) = 0,\ \boldsymbol{Y} \in \mathcal{S}(m)_+ \text{ and } \boldsymbol{X} \bullet \boldsymbol{Y} = 0. \end{array}$$

Choosing $\lambda_i \geq 0$ so as to $f_i(\boldsymbol{X}) + \lambda_i = 0$ for each i $(1 \leq i \leq r)$, we can formulate the KKT condition as an SDCP:

$$\begin{array}{ll} \text{Find} & \boldsymbol{U} = \begin{pmatrix} \boldsymbol{X} & \boldsymbol{O} \\ \boldsymbol{O} & \mathbf{diag}\{\lambda_i\} \end{pmatrix} \succeq \boldsymbol{O},\ \boldsymbol{V} = \begin{pmatrix} \boldsymbol{Y} & \boldsymbol{O} \\ \boldsymbol{O} & \mathbf{diag}\{\mu_i\} \end{pmatrix} \succeq \boldsymbol{O} \\ \text{such that} & (\boldsymbol{U}, \boldsymbol{V}) \in \mathcal{F}^{cp} = \left\{ (\boldsymbol{U}, \boldsymbol{V}) : \begin{array}{l} \boldsymbol{Y} = Df_0(\boldsymbol{X}) + \sum \mu_i Df_i(\boldsymbol{X}) + \sum \nu_j Df_j(\boldsymbol{X}), \\ f_i(\boldsymbol{X}) + \lambda_i = 0,\ f_j(\boldsymbol{X}) = 0 \end{array} \right\}, \\ & \boldsymbol{U} \bullet \boldsymbol{V} = 0 \ (i.e., \boldsymbol{X} \bullet \boldsymbol{Y} = 0,\ \lambda_i \mu_i = 0 \ (1 \leq i \leq r)). \end{array} \tag{3}$$

Maximal monotonicity of $\mathcal{F}^{cp}$ follows from [5, Section 6.7]. The variables ($\boldsymbol{U}$ and $\boldsymbol{V}$) in this SDCP (3) are elements of $\mathcal{S} := \mathcal{S}(m, 1, \cdots, 1) = \mathbf{diag}\{\mathcal{S}(m), \mathcal{R}, \cdots, \mathcal{R}\}$ (for simplicity, we identify $\mathcal{S}(1)$ as $\mathcal{R}$).

Example 1.1. (Log-Likelihood [4, 7]) : The Wishart distribution is an important distribution in many statistical models and applications. The likelihood associated with the distribution needs to be maximized, as a function of its matrix argument $\boldsymbol{\Sigma}$, over the restricted region $\boldsymbol{O} \prec \boldsymbol{L} \preceq \boldsymbol{\Sigma} \preceq \boldsymbol{U}$, where $\boldsymbol{L}$ and $\boldsymbol{U}$ are known symmetric matrices. One example of this type of ordering is when a vector of observations is observed through a filter. Thus, one observes $\underline{Y} = \underline{X} + \underline{E}$, where $\underline{E}$ represents the error introduced by the filter. If by calibration experiments the covariance matrix of $\underline{E}$ was determined to be $\boldsymbol{L}$, then when estimating the covariance matrix of $\underline{Y}$ one would require that $\boldsymbol{L} \preceq \boldsymbol{\Sigma}_Y$.

Let $\boldsymbol{\Sigma}$ be a $k \times k$ symmetric positive definite matrix, and $\boldsymbol{A}$ be a random symmetric $k \times k$ positive definite matrix with a $\text{Wishart}_k(n, \boldsymbol{\Sigma})$ distribution, $n \geq k$. The log-likelihood of $\boldsymbol{\Sigma}$ is

$$l_0(\boldsymbol{\Sigma}; \boldsymbol{A}) = -\frac{n}{2} \log\det \boldsymbol{\Sigma} - \frac{1}{2} \boldsymbol{A} \bullet \boldsymbol{\Sigma}^{-1}$$

plus a constant. By replacing $\boldsymbol{\Lambda} := \boldsymbol{\Sigma}^{-1}$, the problem can be stated as one of minimizing

$$l(\boldsymbol{\Lambda}; \boldsymbol{A}) = -n \log\det \boldsymbol{\Lambda} + \boldsymbol{A} \bullet \boldsymbol{\Lambda}$$

over the region $\boldsymbol{O} \prec \boldsymbol{U}^{-1} \preceq \boldsymbol{\Lambda} \preceq \boldsymbol{L}^{-1}$. This problem is a convex programming problem over the positive semidefinite cone $\mathcal{S}(k)_+$ in $\mathcal{S}(k)$. ∎

Example 1.2. (Fat Triangulation [55]) : In [55], Todd showed the best linear transformation of three types of triangulations under four criteria using first order conditions and a special argument using the generalized singular value decomposition. For example, two of the problems are as follows;

$$\begin{array}{lll} P_1 & \text{minimize} & f(\boldsymbol{A}) = \sum_i \|\boldsymbol{A}^{-\top} \boldsymbol{e}_i\| + \sum_{i<j} \|\boldsymbol{A}^{-\top}(\boldsymbol{e}_i - \boldsymbol{e}_j)\| \\ & \text{subject to} & g_s(\boldsymbol{A}) \leq 1, \\ Q_1 & \text{minimize} & f(\boldsymbol{A}) = -|\det \boldsymbol{A}| \\ & \text{subject to} & g_s(\boldsymbol{A}) \leq 1, \end{array}$$

where $\boldsymbol{e}_i$ is the i-th unit vector and $g_s(\boldsymbol{A}) = \|\boldsymbol{A}\boldsymbol{s}\|$ for all $(0,1)$-vectors $\boldsymbol{s}$. He recently suggested that all these problems can be transformed into convex programming problems over positive semidefinite cones subject to linear constraints and hence the proof given in [55] can be rewritten in very straightforward way using the variable $\boldsymbol{H} := \boldsymbol{A}^\top \boldsymbol{A}$. ∎

We begin by showing the location of our research. Several problems arising in different fields, such as mathematical programming, game theory, economic equilibrium, mechanics, systems and control theory etc., are transformed into a monotone complementarity problem over the nonnegative orthant $\mathcal{R}^n_+$, the positive semidefinite cone $\mathcal{S}_+$ or a general convex cone C [1, 6, 8, 9, 20, 44, 49, etc.].

Definition 1.3. Let $\mathcal{E}$ be a finite dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{E} \times \mathcal{E}$. A set $\mathcal{F} \subset \mathcal{E} \times \mathcal{E}$ is called *monotone* if $\langle \boldsymbol{x} - \boldsymbol{x}', \boldsymbol{y} - \boldsymbol{y}' \rangle \geq 0$ for any $(\boldsymbol{x}, \boldsymbol{y})$, $(\boldsymbol{x}', \boldsymbol{y}') \in \mathcal{F}$. Moreover, the monotone set $\mathcal{F}$ is said to be *maximal* if there is no monotone set in $\mathcal{E} \times \mathcal{E}$ which properly contains $\mathcal{F}$. ∎

A monotone complementarity problem (CP) over the nonnegative orthant has important applications in linear programming (LP), convex quadratic and convex programs (see, *e.g.*, Cottle, Pang and Stone [8], Dantzig and Cottle [9], Karamardian [19] and Mangasarian and McLinden [35]):

$$\begin{array}{lll} \textbf{CP} & \text{Find an} & (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{R}^n_+ \times \mathcal{R}^n_+ \\ & \text{such that} & (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{F}^{CP} \text{ and } \langle \boldsymbol{x}, \boldsymbol{y} \rangle := \boldsymbol{x}^\top \boldsymbol{y} = 0, \end{array} \tag{4}$$

where $\mathcal{F}^{CP}$ is a maximal monotone subset of $\mathcal{R}^n \times \mathcal{R}^n$. Remark that the CP (4) can be naturally embedded in the SDCP (1) in $\mathcal{S} := \mathcal{S}(1, \cdots, 1) = \mathbf{diag}\{\mathcal{R}, \cdots, \mathcal{R}\}$. For the case $\mathcal{F}^{CP}$ is affine, we call the monotone CP (4) the monotone linear complementarity problem (LCP). The monotone LCP has a solution if and only if the feasible set $\mathcal{F}^{LCP}_+ := \mathcal{F}^{LCP} \cap (\mathcal{R}^n_+ \times \mathcal{R}^n_+)$ is nonempty.

Karamardian [20] treated of a monotone generalized complementarity problem (GCP) (or a monotone complementarity problem over a convex cone):

$$\begin{array}{lll} \textbf{GCP} & \text{Find an} & (\boldsymbol{x}, \boldsymbol{y}) \in C \times C^* \\ & \text{such that} & (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{F}^{GCP} \text{ and } \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0, \end{array} \tag{5}$$

where C is a convex cone in a finite dimensional real vector space $\mathcal{E}$, C^* is the dual cone of C and $\mathcal{F}^{GCP}$ is a maximal monotone subset of $\mathcal{E} \times \mathcal{E}$. He showed that, in the case that the set $\mathcal{F}^{GCP}$ is given by a continuous monotone function f, *i.e.*, $\mathcal{F}^{GCP} = \{(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{y} = f(\boldsymbol{x})\}$, the monotone GCP (5) has a solution if there exists a point $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{F}^{GCP}$ such that $\boldsymbol{x} \in C$ and $\boldsymbol{y} \in \mathrm{Int}\, C^*$ [20, Theorem 4.1], where Int K denotes an interior of the set K. Gowda and Seidman [12] treated of a monotone generalized linear complementarity problem (GLCP), *i.e.*, the monotone GCP (5) with $\mathcal{F}^{GCP}$ being affine, and proved some important results with respect to the solvability of the GLCP. For example, the defining cone C is not polyhedral if and only if there exists an affine monotone operator such that the corresponding GLCP is feasible but not solvable [12, Theorem 7.1].

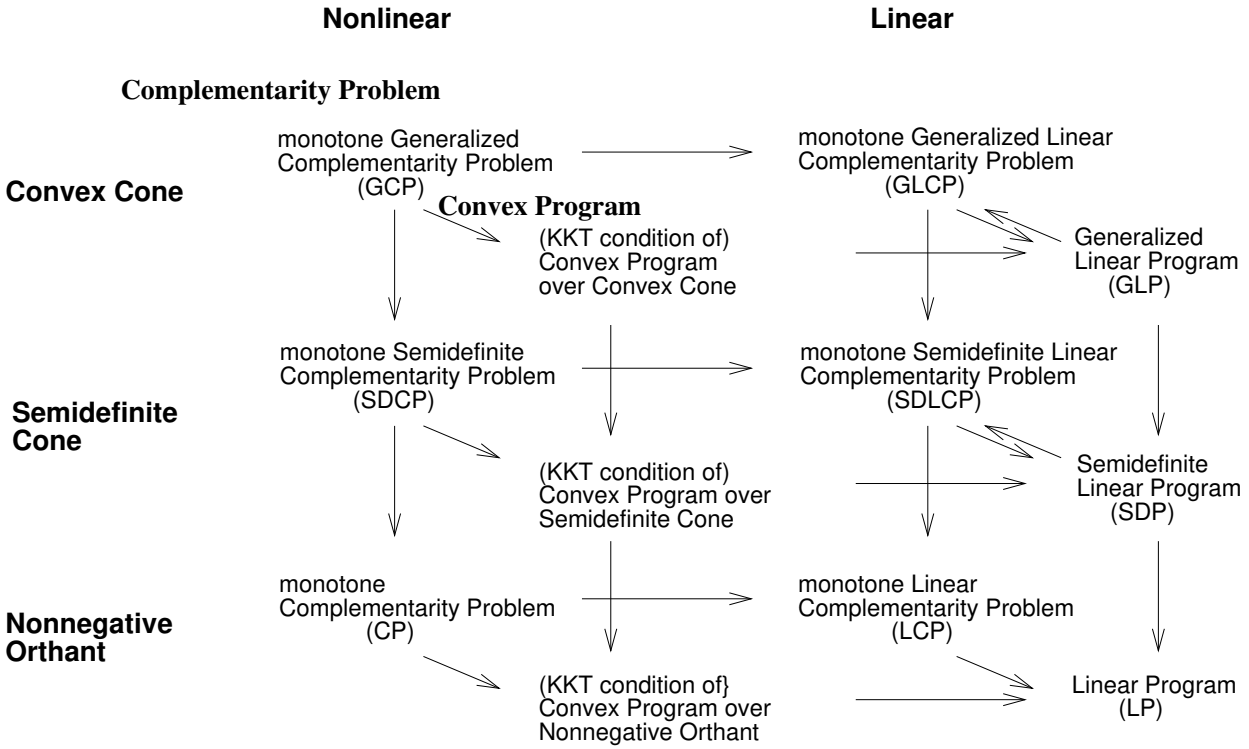

Table 1: Monotone Complementarity Problems and Convex Programs

We also note that the various kind of mathematical program can be transformed into a generalized linear programming (GLP) (a linear programming over a convex cone or linear programming in conic form [44]):

$$\begin{array}{lll} \textbf{GLP} & \text{minimize} & \langle \boldsymbol{c}, \boldsymbol{x} \rangle \\ & \text{subject to} & \boldsymbol{x} \in C \cap [\mathcal{L} + \boldsymbol{b}], \end{array} \tag{6}$$

where $\boldsymbol{c}$ be a vector in $\mathcal{E}$, C is a convex cone in $\mathcal{E}$ and $\mathcal{L}$ is a linear subspace of $\mathcal{E}$.

Many problems in systems and control theory can be formulated as a semidefinite linear program (SDP), *i.e.*, a linear programming over the positive semidefinite cone:

$$\begin{array}{lll} \textbf{SDP} & \text{minimize} & \boldsymbol{C} \bullet \boldsymbol{X} \\ & \text{such that} & \boldsymbol{A}_i \bullet \boldsymbol{X} = b_i \ (1 \leq i \leq m), \\ & & \boldsymbol{X} \succeq \boldsymbol{O}, \ \boldsymbol{X} \in \mathcal{S}, \end{array} \tag{7}$$

where $\boldsymbol{C}$, $\boldsymbol{A}_i$ are in $\mathcal{S}$ and $b_i \in \mathcal{R}$. Many studies for SDP (7) have been done [1, 2, 3, 6, 10, 18, 26, 27, 28, 29, 30, 32, 42, 44, 48, 54, 56, 59, etc]. Kojima et al. [26, 27, 28, 29] also treated of a monotone semidefinite linear complementarity problem (SDLCP) (or a monotone linear complementarity problem over positive semidefinite cone $\mathcal{S}_+$ in $\mathcal{S}$). Kojima, Shida and Shindoh [27] presented the constructive way of reducing a monotone GLCP to a GLP. As a direct consequence of this, the monotone SDLCP is reducible to the SDP (7), see [27].

As we see above, the monotone SDCP (1) includes the monotone CP (4), the SDP (7) and the Convex Programming (2) over the positive semidefinite cone. Thus, the monotone SDCP (1) is one of the important classes of the GCP (5).

In a number of literature on the (generalized) complementarity problems, the maximal monotone set $\mathcal{F}$ is given by a graph of a (point-to-point or point-to-set) mapping T, *i.e.*, $\mathcal{F} = \{(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{y} \in T(\boldsymbol{x})\}$. For such a problem, the solution is the projection of the solution pair $(\boldsymbol{x}, \boldsymbol{y})$ in $\mathcal{F}^*$ to the first $\boldsymbol{x}$-space (let $P_x(\mathcal{F}^*) \equiv \{\boldsymbol{x} : (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{F}^*\}$). However any maximal monotone set $\mathcal{F}$ can be described by a graph of a point-to-set mapping T, the mapping T is not explicitly given in practice, *e.g.*, the primal-dual pair of the (SD)LP ($\mathcal{F} = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{R}^n \times \mathcal{R}^n : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{A}^\top \boldsymbol{\lambda} + \boldsymbol{y} = \boldsymbol{c}$ for some $\boldsymbol{\lambda}\}$) or the Karush-Kuhn-Tucker condition (3) of the convex programming. This is the reason why we consider the maximal monotone subset $\mathcal{F}$ in stead of the maximal monotone map T.

One of the important conceptions in this paper is the trajectory or the path of centers, which has been playing a substantial role in the development of interior-point methods for the LP and the monotone (L)CP [14, 21, 22, 23, 24, 36, 43, etc]. The more fundamental conception, the center was introduced for convex sets by Huard [17] and for polytopes by Sonnevend [52], in connection with optimization problems. It was extended by Megiddo [40] to the trajectory for an LP (linear program) and to the trajectory for a monotone LCP (linear complementarity problem). In his paper [40], Megiddo showed that the trajectory runs through the interior of the primal-dual feasible region of the LP (or the interior of the feasible region of the monotone LCP) and converges to a pair of primal-dual optimal solutions of the LP (or a solution of the LCP, respectively). Since then, numerous interior-point algorithms have been developed based on the idea of "numerically tracing the trajectory." Recently, the conception of the trajectory is extended to more general problems such as the SDP (7) and the monotone SD(L)CP [1, 2, 11, 29, 32, etc], and the GLP (6) and the monotone GCP (5) [15, 44, 45, 46, 50].

The aim of this paper is to show some properties of the trajectory (the path of centers) and the solution set of the monotone SDCP (1). We also present a globally convergent infeasible-interior-point algorithm for the monotone SDCP (1). For simplicity, we use the following notations;

$$\begin{array}{lcl} \mathcal{F}(\boldsymbol{B}) & := & \mathcal{F} + \boldsymbol{B} = \{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S} \times \mathcal{S} : (\boldsymbol{X} - \boldsymbol{B}_X, \boldsymbol{Y} - \boldsymbol{B}_Y) \in \mathcal{F})\}, \\ \mathcal{F}_+(\boldsymbol{B}) & := & \{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}(\boldsymbol{B}) : \boldsymbol{X} \succeq \boldsymbol{O}, \boldsymbol{Y} \succeq \boldsymbol{O}\}, \\ \mathcal{F}_{++}(\boldsymbol{B}) & := & \{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}(\boldsymbol{B}) : \boldsymbol{X} \succ \boldsymbol{O}, \boldsymbol{Y} \succ \boldsymbol{O}\}. \end{array}$$

As in [50], for every $\boldsymbol{a} = (a_1, \cdots, a_l) \in \mathcal{R}^l_{++}$ (weight) and $\boldsymbol{B} = (\boldsymbol{B}_X, \boldsymbol{B}_Y) \in \mathcal{S} \times \mathcal{S}$ (residual), a *center* $\boldsymbol{c}(\boldsymbol{a}, \boldsymbol{B})$ of the monotone SDCP (1) is defined by

$$\boldsymbol{c}(\boldsymbol{a}, \boldsymbol{B}) := \{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}_+(\boldsymbol{B}) : \boldsymbol{X}^q \boldsymbol{Y}^q = a^q \boldsymbol{I}^q,\ q \in \{1, \cdots, l\}\}.$$

By [50], the center $\boldsymbol{c}(\boldsymbol{a}, \boldsymbol{B})$ consists of a single point and is continuous w.r.t. $\mathcal{R}^l_{++} \times \mathcal{B}_{++}$, where $\boldsymbol{B} \in \mathcal{B}_{++} := \{\boldsymbol{B} \in \mathcal{S} \times \mathcal{S} : \mathcal{F}(\boldsymbol{B})_{++} \neq \emptyset\}$. We also consider a "trajectory" for the SDCP (1). Let $\boldsymbol{a}(t), \boldsymbol{B}(t)$ be continuous mappings w.r.t. $t \in [0.1]$ such that

$$\begin{array}{l} \boldsymbol{B}(t) \in \mathcal{B}_{++} \ \text{ for every } \ t \in (0, 1], \\ \boldsymbol{a}(t) \in \mathcal{R}^l_{++} \ \text{ for every } \ t \in (0, 1], \\ \boldsymbol{a}(0) = \boldsymbol{0},\ \boldsymbol{B}(0) = \boldsymbol{O}, \\ (\boldsymbol{a}(t_1), \boldsymbol{B}(t_1)) = (\boldsymbol{a}(t_2), \boldsymbol{B}(t_2)) \Rightarrow t_1 = t_2. \end{array} \tag{8}$$

For such two mapping $\boldsymbol{a}(\cdot),\ \boldsymbol{B}(\cdot)$, denote a *trajectory* by $T = \{\boldsymbol{c}(\boldsymbol{a}(t), \boldsymbol{B}(t)) : t \in (0, 1]\}$. By the continuity of the center $\boldsymbol{c}(\boldsymbol{a}, \boldsymbol{B})$, the trajectory T forms a continuous path. Hence,

combined together Section 2, we numerically trace the trajectory T as $t \to 0$ to solve the monotone SDCP (1). If in addition, $\boldsymbol{a}(t) = t\boldsymbol{a}(1)$ and $\boldsymbol{B}(t) = t\boldsymbol{B}(1)$, we call the set T a "trajectory T^L with a linear continuation".

This paper presents the following:

- The convergence properties of the trajectories of the monotone SDCP (1) (Section 2). The trajectory T continuously accumulates into the solution set $\mathcal{F}^*$ of the monotone SDCP (1) as $t \to 0$ passing through the set of the infeasible but positive definite matrices under a certain condition.
- The basic structure of the solutions of the monotone SDCP (1) (Theorem 3.1).
- Existence of a unique limiting point of the trajectory with a linear continuation of the monotone SDLCP under the strict complementarity condition (Theorem 4.1).
- Globally convergent infeasible-interior point algorithm for the monotone SDCP (1) (Section 5).

Throughout this paper, we use the following symbols;

$$
\begin{aligned}
\mathcal{R}^n &: n\text{-dimensional Euclidean space,}\\
\mathcal{R}^n_+ &= \{\boldsymbol{x} \in \mathcal{R}^n : \ x_i \geq 0 \ \text{ for all } i \in \{1, \cdots, n\}\},\\
\mathcal{R}^n_{++} &= \{\boldsymbol{x} \in \mathcal{R}^n : \ x_i > 0 \ \text{ for all } i \in \{1, \cdots, n\}\},\\
\mathcal{M}(n) &: \text{the class of } n \times n \text{ real matrices,}\\
\mathcal{S}(n) &: \text{the class of } n \times n \text{ symmetric real matrices,}\\
\mathcal{M}(n^1, n^2, \cdots, n^l) &= \mathbf{diag}\{\mathcal{M}(n^1), \cdots, \mathcal{M}(n^l)\},\\
&\quad \text{the class of the block-diagonal matrices,}\\
\mathcal{S} &= \mathcal{S}(n^1, \cdots, n^l) = \mathbf{diag}\{\mathcal{S}(n^1), \cdots, \mathcal{S}(n^l)\},\\
\mathcal{S}_+ &= \{\boldsymbol{X} \in \mathcal{S} : \boldsymbol{X} \succeq \boldsymbol{O}\},\\
\mathcal{S}_{++} &= \{\boldsymbol{X} \in \mathcal{S} : \boldsymbol{X} \succ \boldsymbol{O}\},\\
\mathcal{F}(\boldsymbol{B}) &= \{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S} \times \mathcal{S} : (\boldsymbol{X} - \boldsymbol{B}_X, \boldsymbol{Y} - \boldsymbol{B}_Y) \in \mathcal{F}\},\\
\mathcal{F}_+(\boldsymbol{B}) &= \{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}(\boldsymbol{B}) : \boldsymbol{X} \succeq \boldsymbol{O}, \ \boldsymbol{Y} \succeq \boldsymbol{O}\},\\
\mathcal{F}_{++}(\boldsymbol{B}) &= \{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}(\boldsymbol{B}) : \boldsymbol{X} \succ \boldsymbol{O}, \ \boldsymbol{Y} \succ \boldsymbol{O}\},\\
\mathcal{F}^*(\boldsymbol{B}) &= \{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}_+(\boldsymbol{B}) : \boldsymbol{X} \bullet \boldsymbol{Y} = 0\},\\
\mathcal{F}_+ &= \mathcal{F}_+(\boldsymbol{O}) = \{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F} : \boldsymbol{X} \succeq \boldsymbol{O}, \boldsymbol{Y} \succeq \boldsymbol{O}\}\\
&\quad \text{(the set of feasible solutions of the SDCP (1)),}\\
\mathcal{F}_{++} &= \mathcal{F}_{++}(\boldsymbol{O}) = \{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F} : \boldsymbol{X} \succ \boldsymbol{O}, \ \boldsymbol{Y} \succ \boldsymbol{O}\}\\
&\quad \text{(the set of interior feasible solutions of the SDCP (1)),}\\
\mathcal{F}^* &= \mathcal{F}^*(\boldsymbol{O}) = \{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}_+ : \boldsymbol{X} \bullet \boldsymbol{Y} = 0\}\\
&\quad \text{(the set of solutions of the SDCP (1)),}\\
\mathcal{B}_+ &= \{\boldsymbol{B} \in \mathcal{S} \times \mathcal{S} : \mathcal{F}_+(\boldsymbol{B}) \neq \emptyset\},\\
\mathcal{B}_{++} &= \{\boldsymbol{B} \in \mathcal{S} \times \mathcal{S} : \mathcal{F}_{++}(\boldsymbol{B}) \neq \emptyset\},\\
\boldsymbol{c}(\boldsymbol{a}, \boldsymbol{B}) &= \{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}_+(\boldsymbol{B}) : \boldsymbol{X}^q \boldsymbol{Y}^q = a^q \boldsymbol{I}^q, q \in \{1, \cdots, l\}\} \quad \text{(a "center"),}
\end{aligned}
$$

$$
\begin{aligned}
T &= \{\boldsymbol{c}(\boldsymbol{a}(t), \boldsymbol{B}(t)) : t \in (0,1], \boldsymbol{a}(t) \text{ and } \boldsymbol{B}(t) \text{ as (8)}\} \quad \text{(a "trajectory")}, \\
T^L &= \{\boldsymbol{c}(\boldsymbol{a}(t), \boldsymbol{B}(t)) : \boldsymbol{a}(t) = t\boldsymbol{a}(1), \boldsymbol{B}(t) = t\boldsymbol{B}(1), t \in (0,1]\} \\
&\quad \text{(a "trajectory with a linear continuation")},
\end{aligned}
$$

where $\boldsymbol{B} = (\boldsymbol{B}_X, \boldsymbol{B}_Y) \in \mathcal{S} \times \mathcal{S}$, $\boldsymbol{a} \in \mathcal{R}^l_+$, $\boldsymbol{I}^q$ is an $(n^q \times n^q)$-identity matrix and $\boldsymbol{O}$ is a zero matrix in an appropriate size..

2 Trajectories.

In this section, we state the convergence properties of the trajectory T of the monotone SDCP (1). Let

$$\underline{t} = \inf\{t \in (0,1] : (\boldsymbol{a}(t), \boldsymbol{B}(t)) \in \mathcal{R}^l_{++} \times \mathcal{B}_{++}\}.$$

By [50], the trajectory T forms a continuous path on $t \in [\underline{t}, 1]$. Moreover, if $\underline{t} = 0$, the trajectory accumulates into the solution set $\mathcal{F}^*$ as $t \to 0$. Under certain conditions (see Theorems 2.2, 2.3, 2.7 and Corollary 2.6), we have $\underline{t} = 0$. Therefore, to solve the monotone SDCP (1), we numerically trace the trajectory T as $t \to 0$.

The convexity of $\mathcal{B}_{++}$ plays a fundamental role when we consider the trajectory (with a linear continuation, especially).

Lemma 2.1.

(i) *$\mathcal{B}_{++}$ is a nonempty open convex subset of $\mathcal{S} \times \mathcal{S}$.*

(ii) *$\mathcal{B}_{++} \subset \mathcal{B}^* \subset \mathcal{B}_+ \subset cl\mathcal{B}_{++}$, where $cl\mathcal{B}_{++}$ denotes the closure of $\mathcal{B}_{++}$ and $\mathcal{B}^* = \{\boldsymbol{B} \in \mathcal{S} \times \mathcal{S} : \mathcal{F}^*(\boldsymbol{B}) \neq \emptyset\}$.* ∎

Note that if $\boldsymbol{B} \in \mathcal{B}_{++}$ then $\boldsymbol{B} + (\mathcal{S}_+ \times \mathcal{S}_+) \subset \mathcal{B}_{++}$, *i.e.*, the recession cone of $\mathcal{B}$ contains $\mathcal{S}_+ \times \mathcal{S}_+$.

The following theorems are extension of the results obtained in [22, Section 3] for the monotone CP (4). All proofs are based on [22, Section 3].

Theorem 2.2. *Suppose that the trajectory T is bounded. Then,*

(i) *the trajectory T has at least one accumulation point as $t \to \underline{t} = 0$,*

(ii) *if $(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}})$ is an accumulation point of the trajectory T as $t \to 0$, then $(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}})$ is a solution of the SDCP (1).* ∎

Proof: Obviously, the trajectory T has an accumulation point as $t \to \underline{t}$. Suppose that $(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}}, \underline{t})$ is an accumulation point of T. Then there is a sequence $\{t^k \in (\underline{t}, 1]\}$ such that

$$\lim_{k\to\infty} \left(\boldsymbol{c}(\boldsymbol{a}(t^k), \boldsymbol{B}(t^k)), \boldsymbol{t}^k\right) = (\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}}, \underline{t}) \in \mathcal{S}_+ \times \mathcal{S}_+ \times [0,1].$$

Then we have $(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}}) \in \mathcal{F}_+(\boldsymbol{B}(\underline{t}))$. We shall assume that $\underline{t} > 0$, and derive a contradiction. Since $\underline{t} > 0$, we have $\boldsymbol{a}(\underline{t}) \in \mathcal{R}^l_{++}$, and then $(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}}) \in \mathcal{S}_{++} \times \mathcal{S}_{++}$. Therefore, we have $\boldsymbol{B}(\underline{t}) \in \mathcal{B}_{++}$. By Lemma 2.1, the set $\mathcal{B}_{++}$ is open. Hence, by the continuity, we have

$$\boldsymbol{a}(\underline{t} - \epsilon) \in \mathcal{R}^l_{++}, \ \boldsymbol{B}(\underline{t} - \epsilon) \in \mathcal{B}_{++}$$

for every sufficiently small $\epsilon > 0$. This contradicts to the construction of $\underline{t}$. Thus we have $\underline{t} = 0$. By the continuity of the inner product and the construction of the mappings $\boldsymbol{a}(t)$ and $\boldsymbol{B}(t)$, we conclude that $(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}})$ is a solution of the monotone SDCP (1). ∎

Theorem 2.2 ensures that tracing the trajectory T leads to solutions of the SDCP (1) whenever T is bounded. However, the trajectory T does not necessarily converge to a single point $(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}})$ as $t \to 0$, *i.e.*, there may be more than one accumulation point.

Next theorem guarantees that the trajectory T forms a bounded set.

Theorem 2.3. *Assume that $\boldsymbol{B}(t) \in \mathcal{B}_{++}$ for all $t \in [0, 1]$. Then, the trajectory T is bounded and $\underline{t} = 0$.*

To prove Theorem 2.3, we need the following Lemmas.

Lemma 2.4. *Let $\boldsymbol{a} \in \mathcal{R}^l_{++}$ and $\boldsymbol{B} \in \mathcal{B}_{++}$. Assume that $(\hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}}) \in \mathcal{F}_+$ and $(\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{c}(\boldsymbol{a}, \boldsymbol{B})$. Then*

$$(\hat{\boldsymbol{X}} + \boldsymbol{B}_X) \bullet \boldsymbol{Y} + \boldsymbol{X} \bullet (\hat{\boldsymbol{Y}} + \boldsymbol{B}_Y) \leq \hat{\boldsymbol{X}} \bullet \hat{\boldsymbol{Y}} + \sum_{q=1}^{l} a^q n^q + \hat{\boldsymbol{X}} \bullet \boldsymbol{B}_Y + \boldsymbol{B}_X \bullet \hat{\boldsymbol{Y}} - \boldsymbol{B}_X \bullet \boldsymbol{B}_Y.$$

Proof: By the assumption, there is a point $(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}}) := (\boldsymbol{X}, \boldsymbol{Y}) - (\boldsymbol{B}_X, \boldsymbol{B}_Y)$ in $\mathcal{F}$, *i.e.*, $(\boldsymbol{X}, \boldsymbol{Y}) = (\bar{\boldsymbol{X}} + \boldsymbol{B}_X, \bar{\boldsymbol{Y}} + \boldsymbol{B}_Y)$. Then we have

$$\begin{aligned}
\hat{\boldsymbol{X}} \bullet \boldsymbol{Y} + \boldsymbol{X} \bullet \hat{\boldsymbol{Y}} &= \hat{\boldsymbol{X}} \bullet \hat{\boldsymbol{Y}} + \boldsymbol{X} \bullet \boldsymbol{Y} - (\boldsymbol{X} - \hat{\boldsymbol{X}}) \bullet (\boldsymbol{Y} - \hat{\boldsymbol{Y}}) \\
&= \hat{\boldsymbol{X}} \bullet \hat{\boldsymbol{Y}} + \sum_{q=1}^{l} a^q n^q - (\bar{\boldsymbol{X}} + \boldsymbol{B}_X - \hat{\boldsymbol{X}}) \bullet (\bar{\boldsymbol{Y}} + \boldsymbol{B}_Y - \hat{\boldsymbol{Y}}) \\
&\leq \hat{\boldsymbol{X}} \bullet \hat{\boldsymbol{Y}} + \sum_{q=1}^{l} a^q n^q - [\boldsymbol{B}_X \bullet (\boldsymbol{Y} - \hat{\boldsymbol{Y}}) + (\boldsymbol{X} - \hat{\boldsymbol{X}}) \bullet \boldsymbol{B}_Y + \boldsymbol{B}_X \bullet \boldsymbol{B}_Y] \\
&= \hat{\boldsymbol{X}} \bullet \hat{\boldsymbol{Y}} + \sum_{q=1}^{l} a^q n^q + \boldsymbol{B}_X \bullet \hat{\boldsymbol{Y}} + \hat{\boldsymbol{X}} \bullet \boldsymbol{B}_Y - \boldsymbol{B}_X \bullet \boldsymbol{Y} - \boldsymbol{X} \bullet \boldsymbol{B}_Y - \boldsymbol{B}_X \bullet \boldsymbol{B}_Y.
\end{aligned}$$

The inequality follows from the monotonicity of $\mathcal{F}$. ∎

Lemma 2.5. *Suppose that $\boldsymbol{Y}_0 \in \mathcal{S}_{++}$. For any fixed constant α, the set $\{\boldsymbol{X} \in \mathcal{S}_+ : \boldsymbol{X} \bullet \boldsymbol{Y}_0 \leq \alpha\}$ is bounded.*

Proof: If $\alpha < 0$, the set $\{\boldsymbol{X} \in \mathcal{S}_+ : \boldsymbol{X} \bullet \boldsymbol{Y}_0 \leq \alpha\}$ is empty since $\boldsymbol{Y}_0$ lies in $\mathcal{S}_{++}$. Assume that $\alpha \geq 0$. Then $\boldsymbol{O} \in \{\boldsymbol{X} \in \mathcal{S}_+ : \boldsymbol{X} \bullet \boldsymbol{Y}_0 \leq \alpha\}$. Assume that the set $\{\boldsymbol{X} \in \mathcal{S} : \boldsymbol{X} \bullet \boldsymbol{Y}_0 \leq \alpha\}$ is unbounded. Since the set $\{\boldsymbol{X} \in \mathcal{S}_+ : \boldsymbol{X} \bullet \boldsymbol{Y}_0 \leq \alpha\}$ is convex, there is an unbounded ray $t\boldsymbol{V}$ $(t \geq 0)$ such that $\boldsymbol{V} \neq \boldsymbol{O}$ and $t\boldsymbol{V} \in \{\boldsymbol{X} \in \mathcal{S}_+ : \boldsymbol{X} \bullet \boldsymbol{Y}_0 \leq \alpha\}$ for all $t \geq 0$. This implies that $\boldsymbol{V} \in \mathcal{S}_+ \backslash \{\boldsymbol{O}\}$ and $\boldsymbol{V} \bullet \boldsymbol{Y}_0 = 0$. Since $\boldsymbol{Y}_0 \in \mathcal{S}_{++}$, there exists a small positive $\epsilon > 0$ such that $\boldsymbol{Y}_0 - \epsilon \boldsymbol{V} \in \mathcal{S}_+$. Thus,

$$\begin{array}{rcll} 0 & \leq & \boldsymbol{V} \bullet (\boldsymbol{Y}_0 - \epsilon \boldsymbol{V}) & (\text{since } \boldsymbol{V} \in \mathcal{S}_+ \text{ and } \boldsymbol{Y}_0 - \epsilon \boldsymbol{V} \in \mathcal{S}_+) \\ & = & -\epsilon \boldsymbol{V} \bullet \boldsymbol{V} & (\text{since } \boldsymbol{V} \bullet \boldsymbol{Y}_0 = 0) \\ & < & 0. & \end{array}$$

This is a contradiction. Therefore, the set $\{\boldsymbol{X} \in \mathcal{S}_+ : \boldsymbol{X} \bullet \boldsymbol{Y}_0 \leq \alpha\}$ is bounded. ∎

Proof of Theorem 2.3: By the assumption, we have that $\underline{t} = 0$ and $\mathcal{F}_{++} \neq \emptyset$ (let $(\hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}}) \in \mathcal{F}_{++}$). To prove the boundedness of the trajectory T, assume on the contrary that T is unbounded. Then there is a sequence $\{t^k \in (0, 1]\}$ such that

$$\lim_{k \to \infty} t^k = 0, \quad \lim_{k \to \infty} \|(\boldsymbol{X}(\boldsymbol{t}^k), \boldsymbol{Y}(t^k))\| = \infty.$$

By Lemma 2.5, we have

$$\lim_{k \to \infty} \hat{\boldsymbol{X}} \bullet \boldsymbol{Y}(t^k) + \boldsymbol{X}(t^k) \bullet \hat{\boldsymbol{Y}} = \infty.$$

For sufficiently large k, each $(\boldsymbol{X}(t^k), \boldsymbol{Y}(t^k))$ satisfies the inequality

$$\begin{aligned} \frac{1}{2}[\hat{\boldsymbol{X}} \bullet \boldsymbol{Y}(t^k) + \boldsymbol{X}(t^k) \bullet \hat{\boldsymbol{Y}}] &\leq (\hat{\boldsymbol{X}} + \boldsymbol{B}_X(t^k)) \bullet \boldsymbol{Y}(t^k) + \boldsymbol{X}(t^k) \bullet (\hat{\boldsymbol{Y}} + \boldsymbol{B}_X(t^k)) \\ &\qquad (\text{since } \boldsymbol{B}(t^k) \to \boldsymbol{0} \text{ as } k \to \infty \text{ and } (\hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}}) \in \mathcal{S}_{++} \times \mathcal{S}_{++}) \\ &\leq \hat{\boldsymbol{X}} \bullet \hat{\boldsymbol{Y}} + \sum_{q=1}^{l} a^q(t^k) n^q + \hat{\boldsymbol{X}} \bullet \boldsymbol{B}_Y(t^k) + \boldsymbol{B}_X(t^k) \bullet \hat{\boldsymbol{Y}} - \boldsymbol{B}_X(t^k) \bullet \boldsymbol{B}_Y(t^k) \\ &\qquad (\text{by Lemma 2.4}). \end{aligned}$$

The first line above diverges toward ∞ as $k \to \infty$, while the right-hand side of the last line converges to $\hat{\boldsymbol{X}} \bullet \hat{\boldsymbol{Y}}$. This is a contradiction. ∎

Now we are ready to state a linear continuation of the SDCP (1) under some assumptions. Corollary 2.6 is a direct consequence of Lemma 2.1 and Theorem 2.3.

Corollary 2.6. *Let $\bar{\boldsymbol{a}} \in \mathcal{R}^l_{++}$, $\bar{\boldsymbol{B}} = (\bar{\boldsymbol{B}}_X, \bar{\boldsymbol{B}}_Y) \in \mathcal{B}_{++}$ and $\boldsymbol{O} = (\boldsymbol{O}_X, \boldsymbol{O}_Y) \in \mathcal{B}_{++}$. Let $\boldsymbol{a}(t) = t\bar{\boldsymbol{a}}$ and $\boldsymbol{B}(t) = t\bar{\boldsymbol{B}}$ for all $t \in [0, 1]$. Then, the trajectory T^L with the linear continuation is bounded and $\underline{t} = 0$.* ∎

Theorem 2.7. *Let $\bar{\boldsymbol{a}} \in \mathcal{R}^l_{++}$ and $\bar{\boldsymbol{B}} = (\bar{\boldsymbol{B}}_X, \bar{\boldsymbol{B}}_Y) \in \mathcal{B}_{++} \cap (\mathcal{S}_{++} \times \mathcal{S}_{++})$. Let $\boldsymbol{a}(t)$, $\boldsymbol{B}(t)$ be as in Corollary 2.6. Then, $\underline{t} = 0$ and the trajectory T^L with the linear continuation is bounded if and only if the SDCP (1) has a solution.*

Proof: By Theorem 2.2, if the trajectory T^L is bounded, then $\mathcal{F}^*$ is nonempty. Assume that the SDCP (1) has a solution $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$. By Lemma 2.4, we see that for all $(\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{c}(\boldsymbol{a}(t), \boldsymbol{B}(t)) \in T^L$,

$$\begin{aligned}(\boldsymbol{X}^* + t\bar{\boldsymbol{B}}_X) \bullet \boldsymbol{Y} + \boldsymbol{X} \bullet (\boldsymbol{Y}^* + t\bar{\boldsymbol{B}}_Y) \\ &\leq \boldsymbol{X}^* \bullet \boldsymbol{Y}^* + t[\sum_{q=1}^{l} \bar{a}^q n^q + \boldsymbol{X}^* \bullet \bar{\boldsymbol{B}}_Y + \bar{\boldsymbol{B}}_X \bullet \boldsymbol{Y}^*] - t^2 \bar{\boldsymbol{B}}_X \bullet \bar{\boldsymbol{B}}_Y \\ &\leq t[\sum_{q=1}^{l} \bar{a}^q n^q + \boldsymbol{X}^* \bullet \bar{\boldsymbol{B}}_Y + \bar{\boldsymbol{B}}_X \bullet \boldsymbol{Y}^*].\end{aligned}$$

Then, we have

$$\bar{\boldsymbol{B}}_X \bullet \boldsymbol{Y} + \boldsymbol{X} \bullet \bar{\boldsymbol{B}}_Y \leq \sum_{q=1}^{l} \bar{a}^q n^q + \boldsymbol{X}^* \bullet \bar{\boldsymbol{B}}_Y + \bar{\boldsymbol{B}}_X \bullet \boldsymbol{Y}^* (=: d) \quad \text{for all } (\boldsymbol{X}, \boldsymbol{Y}) \in T^L.$$

Therefore, if $(\boldsymbol{X}, \boldsymbol{Y}) \in T^L$, then it is contained in the set

$$\{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_+ \times \mathcal{S}_+ : \bar{\boldsymbol{B}}_X \bullet \boldsymbol{Y} + \boldsymbol{X} \bullet \bar{\boldsymbol{B}}_Y \leq d\},$$

which is bounded by Lemma 2.5 and the assumption $(\bar{\boldsymbol{B}}_X, \bar{\boldsymbol{B}}_Y) \in \mathcal{S}_{++} \times \mathcal{S}_{++}$. ∎

3 Solution of Semidefinite Complementarity Problems.

The aim of this section is to discuss the structure of the solution set $\mathcal{F}^*$ of the SDCP (1) (Theorem 3.1).

Theorem 3.1 (ii) is an extension of Kojima et al. [29, Theorem 6.1], in which they treated of the monotone semidefinite complementarity problem (SDLCP). For simplicity, we say that 0×0-matrix is nonsingular.

Theorem 3.1.

(i) *The solution set $\mathcal{F}^*$ is a closed convex set. Moreover, if $\mathcal{F}_{++}$ is nonempty, then $\mathcal{F}^*$ is nonempty and bounded.*

(ii) *There exist an orthogonal matrix $\boldsymbol{P} := \mathbf{diag}\{\boldsymbol{P}^1, \cdots, \boldsymbol{P}^l\}$ and disjoint index subsets I^q, J^q, K^q of $\{1, 2, \cdots, n^q\}$ $(q \in \{1, \cdots, l\})$ such that*

$$\left.\begin{array}{l} I^q \cup J^q \cup K^q = \{1, 2, \cdots, n^q\} \\ [(\boldsymbol{P}^q)^\top \boldsymbol{X}^q \boldsymbol{P}^q]_{ij} = 0 \ (i \in I^q \cup K^q \ \textit{or} \ j \in I^q \cup K^q), \\ [(\boldsymbol{P}^q)^\top \boldsymbol{Y}^q \boldsymbol{P}^q]_{ij} = 0 \ (i \in J^q \cup K^q \ \textit{or} \ j \in J^q \cup K^q) \end{array}\right\} \textit{ for every } (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}^*.$$

Furthermore, if $\mathcal{F}^*$ is nonempty, there exists $(\hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}}) \in \mathcal{F}^*$ such that $[(\boldsymbol{P}^q)^\top \hat{\boldsymbol{X}}^q \boldsymbol{P}^q]_{J^q}$ and $[(\boldsymbol{P}^q)^\top \hat{\boldsymbol{Y}}^q \boldsymbol{P}^q]_{I^q}$ are both nonsingular for every $q \in \{1, \cdots l\}$. Here, $[\boldsymbol{X}]_I$ denotes the $|I| \times |I|$-submatrix of $\boldsymbol{X}$ induced by those rows and columns of $\boldsymbol{X}$ whose indices belong to I.

Before proving Theorem 3.1, we make a remark and a comment.

Remark 3.2. In a number of literature on the (generalized) complementarity problems, the monotone set $\mathcal{F}$ is given by a graph of a (point-to-point or point-to-set) maximal monotone mapping T, *i.e.*, $\mathcal{F} = \{(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{y} \in T(\boldsymbol{x})\}$. For such a problem, the solution is the projection of the solution pair $(\boldsymbol{x}, \boldsymbol{y})$ in $\mathcal{F}^*$ to the first $\boldsymbol{x}$-space (let $P_x(\mathcal{F}^*) \equiv \{\boldsymbol{x} : (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{F}^*\}$). The closedness and convexity of $P_x(\mathcal{F}^*)$ and the compactness of $P_x(\mathcal{F}^*)$ under the condition $\mathcal{F}_{++} \neq \emptyset$ were essentially proved in [37, 53] for more general situation (the monotone variational inequalities). See also [16, 41, 53, 57].

However any maximal monotone set $\mathcal{F}$ can be described by a graph of a (point-to-set) maximal monotone mapping T, the mapping T is not explicitly given in practice. This is the reason why we consider $\mathcal{F}$ and $\mathcal{F}^*$ in stead of T and $P_x(\mathcal{F}^*)$. Güler [14, Lemma 2.3] proved the assertion (i) for the monotone complementarity problem (CPs). The assertion of (i) of Theorem 3.1 is a properties of the monotone generalized complementarity problems (GCPs), see also [50]. ∎

By Theorem 3.1, the nonemptiness of the interior feasible set ($\mathcal{F}_{++} \neq \emptyset$) implies the solvability ($\mathcal{F}^* \neq \emptyset$). For the monotone linear complementarity problem (LCP), the feasibility ($\mathcal{F}_+ \neq \emptyset$) implies the solvability [8, 12]. In fact, Gowda and Seidman [12, Theorem 3.1] showed that, for the monotone generalized "linear" complementarity problem (GLCP) over the "polyhedral" cone, the feasibility is a necessary and sufficient condition of the solvability. However, for the monotone generalized (linear) complementarity problems over the "nonpolyhedral" cone, the feasibility does not imply the solvability. They also showed that the defining cone $C \subset \mathcal{E}$ is not polyhedral if and only if there exists a maximal monotone affine set $\mathcal{F} \subset \mathcal{E} \times \mathcal{E}$ such that the corresponding monotone GLCP is feasible but not solvable [12, Theorem 7.1]. See, for interesting examples, [10, 12, 39, 56],

To prove the assertion (i) of Theorem 3.1, we need the following lemma.

Lemma 3.3 (Proposition 6.7.2 of [5]) : *$\mathcal{F}$ is a maximal monotone set in $\mathcal{S} \times \mathcal{S}$ if and only if*

$$(\boldsymbol{X} - \boldsymbol{U}) \bullet (\boldsymbol{Y} - \boldsymbol{V}) \geq 0 \text{ for all } (\boldsymbol{U}, \boldsymbol{V}) \in \mathcal{F} \Leftrightarrow (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}.$$

∎

Proof of Theorem 3.1:

Part (i) The proof is based on [14, Lemma 2.3], see also [50]. Closedness of the solution set $\mathcal{F}^*$ is easily derived by the fact that $\mathcal{F}^*$ is an intersection of two closed sets $\mathcal{F}$ and $\{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_+ \times \mathcal{S}_+ : \boldsymbol{X} \bullet \boldsymbol{Y} = 0\}$.

We shall show the convexity of $\mathcal{F}^*$. Letting $(\boldsymbol{X}_1, \boldsymbol{Y}_1)$, $(\boldsymbol{X}_2, \boldsymbol{Y}_2)$ be solutions of the monotone SDCP (1) and $0 \leq \lambda \leq 1$, we shall show that $(\lambda \boldsymbol{X}_1 + (1-\lambda)\boldsymbol{X}_2, \lambda \boldsymbol{Y}_1 + (1-\lambda)\boldsymbol{Y}_2)$ is a solution of the the SDCP (1). Since $(\boldsymbol{X}_1, \boldsymbol{Y}_1)$ and $(\boldsymbol{X}_2, \boldsymbol{Y}_2)$ both lie in the convex set $\mathcal{S}_+ \times \mathcal{S}_+$, we first observe the positive semidefiniteness;

$$(\lambda \boldsymbol{X}_1 + (1-\lambda)\boldsymbol{X}_2, \lambda \boldsymbol{Y}_1 + (1-\lambda)\boldsymbol{Y}_2) \in \mathcal{S}_+ \times \mathcal{S}_+. \tag{9}$$

It follows that

$$\begin{aligned} 0 &\leq (\lambda \boldsymbol{X}_1 + (1-\lambda)\boldsymbol{X}_2) \bullet (\lambda \boldsymbol{Y}_1 + (1-\lambda)\boldsymbol{Y}_2) \\ &= -\lambda(1-\lambda)(\boldsymbol{X}_2 - \boldsymbol{X}_1) \bullet (\boldsymbol{Y}_2 - \boldsymbol{Y}_1) && (\text{since } \boldsymbol{X}_1 \bullet \boldsymbol{Y}_1 = \boldsymbol{X}_2 \bullet \boldsymbol{Y}_2 = 0) \\ &\leq 0 && (\text{by the monotonicity of } \mathcal{F}). \end{aligned}$$

Hence we have shown the complementarity;

$$(\lambda \boldsymbol{X}_1 + (1-\lambda)\boldsymbol{X}_2) \bullet (\lambda \boldsymbol{Y}_1 + (1-\lambda)\boldsymbol{Y}_2) = 0. \tag{10}$$

To show the feasibility;

$$(\lambda \boldsymbol{X}_1 + (1-\lambda)\boldsymbol{X}_2, \lambda \boldsymbol{Y}_1 + (1-\lambda)\boldsymbol{Y}_2) \in \mathcal{F}, \tag{11}$$

we apply Lemma 3.3 with $(\boldsymbol{X}, \boldsymbol{Y}) = (\lambda \boldsymbol{X}_1 + (1-\lambda)\boldsymbol{X}_2, \lambda \boldsymbol{Y}_1 + (1-\lambda)\boldsymbol{Y}_2)$. Let $(\boldsymbol{U}, \boldsymbol{V}) \in \mathcal{F}$. Then we see that

$$\begin{aligned} &((\lambda \boldsymbol{X}_1 + (1-\lambda)\boldsymbol{X}_2) - \boldsymbol{U}) \bullet ((\lambda \boldsymbol{Y}_1 + (1-\lambda)\boldsymbol{Y}_2) - \boldsymbol{V}) \\ &= -(\lambda \boldsymbol{X}_1 + (1-\lambda)\boldsymbol{X}_2) \bullet \boldsymbol{V} - \boldsymbol{U} \bullet (\lambda \boldsymbol{Y}_1 + (1-\lambda)\boldsymbol{Y}_2) + \boldsymbol{U} \bullet \boldsymbol{V} \quad (\text{by } (10)) \\ &= \lambda(\boldsymbol{X}_1 - \boldsymbol{U}) \bullet (\boldsymbol{Y}_1 - \boldsymbol{V}) + (1-\lambda)(\boldsymbol{X}_2 - \boldsymbol{U}) \bullet (\boldsymbol{Y}_2 - \boldsymbol{V}) \\ &\qquad (\text{since } \boldsymbol{X}_1 \bullet \boldsymbol{Y}_1 = \boldsymbol{X}_2 \bullet \boldsymbol{Y}_2 = 0) \\ &\geq 0 \qquad (\text{by the monotonicity of } \mathcal{F}). \end{aligned}$$

Thus we have shown (11). By (9), (10) and (11), we can conclude that, for all $\lambda \in [0,1]$, $(\lambda \boldsymbol{X}_1 + (1-\lambda)\boldsymbol{X}_2, \lambda \boldsymbol{Y}_1 + (1-\lambda)\boldsymbol{Y}_2)$ is a solution of the SDCP (1). Therefore, the set $\mathcal{F}^*$ is convex.

Assume that $\mathcal{F}_{++}$ is nonempty ($\exists (\boldsymbol{X}', \boldsymbol{Y}') \in \mathcal{F}_{++}$). Then the solution set $\mathcal{F}^*$ is nonempty by Theorems 2.2 and 2.3 with $\boldsymbol{B}(t) \equiv \boldsymbol{O}$ for all $t \in [0,1]$. To complete the proof, we only have to show that $\mathcal{F}^*$ is contained in a bounded set. Set $\boldsymbol{X}' \bullet \boldsymbol{Y}' = d$. Suppose that $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}^*$. Since $\mathcal{F}$ is monotone and $\boldsymbol{X} \bullet \boldsymbol{Y} = 0$, we have

$$\boldsymbol{X} \bullet \boldsymbol{Y}' + \boldsymbol{X}' \bullet \boldsymbol{Y} \leq \boldsymbol{X}' \bullet \boldsymbol{Y}' + \boldsymbol{X} \bullet \boldsymbol{Y} = d.$$

Since $\boldsymbol{X}' \in \mathcal{S}_{++}$, $\boldsymbol{Y}' \in \mathcal{S}_{++}$, Lemma 2.5 ensures that the set

$$\{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_+ \times \mathcal{S}_+ : \boldsymbol{X} \bullet \boldsymbol{Y}' + \boldsymbol{X}' \bullet \boldsymbol{Y} \leq d\}$$

is bounded.

Part (ii) For the sake of simplicity, we prove the assertion in the case $l = 1$. The following proof is based on that of Theorem 3.5 of [29].

Assume that $\mathcal{F}^* \neq \emptyset$. By (i), the solution set $\mathcal{F}^*$ is a closed convex set. Let s be the dimension of the affine subspace spanned by $\mathcal{F}^*$. Then there exist $s+1$ pairs of affinely independent matrices $(\boldsymbol{X}_0, \boldsymbol{Y}_0), (\boldsymbol{X}_1, \boldsymbol{Y}_1), \cdots, (\boldsymbol{X}_s, \boldsymbol{Y}_s) \in \mathcal{F}^*$. Define

$$\begin{aligned} (\check{\boldsymbol{X}}, \check{\boldsymbol{Y}}) &= \frac{1}{s+1} \sum_{j=0}^{s} (\boldsymbol{X}_j, \boldsymbol{Y}_j), \\ W &= \{\boldsymbol{p} \in \mathcal{R}^n : \boldsymbol{p}^\top \check{\boldsymbol{X}} \boldsymbol{p} = \boldsymbol{p}^\top \check{\boldsymbol{Y}} \boldsymbol{p} = 0\}, \\ U &= \{\boldsymbol{p} \in W^\perp : \boldsymbol{p}^\top \check{\boldsymbol{X}} \boldsymbol{p} = 0\}, \\ V &= \{\boldsymbol{p} \in W^\perp : \boldsymbol{p}^\top \check{\boldsymbol{Y}} \boldsymbol{p} = 0\}, \end{aligned}$$

where $W^\perp$ is an orthogonal complement of W in $\mathcal{R}^n$. Note that $U\dot{+}W$ ($V\dot{+}W$, respectively), the direct sum of the orthogonal linear spaces U and W (V and W, resp.), is the linear subspace spanned by all the eigenvectors of $\check{\boldsymbol{X}}$ ($\check{\boldsymbol{Y}}$, resp.) associated with its zero eigenvalue. Then, for every $\boldsymbol{p} \in U\dot{+}W$,

$$0 = \boldsymbol{p}^\top \check{\boldsymbol{X}} \boldsymbol{p} = \frac{1}{s+1} \sum_{j=0}^{s} \boldsymbol{p}^\top \boldsymbol{X}_j \boldsymbol{p}.$$

Since each $\boldsymbol{X}_j$ is positive semidefinite, we have $\boldsymbol{X}_j \boldsymbol{p} = \boldsymbol{0}$ ($j = 0, 1, \cdots, s$). Hence

$$\boldsymbol{X}\boldsymbol{p} = \boldsymbol{0} \text{ for every } (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}^* \text{ and every } \boldsymbol{p} \in U\dot{+}W. \tag{12}$$

Similarly we have

$$\boldsymbol{Y}\boldsymbol{p} = \boldsymbol{0} \text{ for every } (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}^* \text{ and every } \boldsymbol{p} \in V\dot{+}W. \tag{13}$$

Let $\check{\boldsymbol{p}}_1, \cdots, \check{\boldsymbol{p}}_n$ form a basis of $\mathcal{R}^n$ such that

$\check{\boldsymbol{p}}_1, \cdots, \check{\boldsymbol{p}}_a$	denote the eigenvectors of $\check{\boldsymbol{X}}$ corresponding to nonzero eigenvalues (let $\xi_1, \cdots, \xi_a$ be the corresponding eigenvalues),
$\check{\boldsymbol{p}}_{a+1}, \cdots, \check{\boldsymbol{p}}_b$	form the basis of W, which are the common eigenvectors of $\check{\boldsymbol{X}}$ and $\check{\boldsymbol{Y}}$ with zero eigenvalues,
$\check{\boldsymbol{p}}_{b+1}, \cdots, \check{\boldsymbol{p}}_n$	denote the basis of U, which are the eigenvectors of $\check{\boldsymbol{X}}$ with zero eigenvalues which are orthogonal to W.

We may assume that $\check{\boldsymbol{p}}_1, \cdots, \check{\boldsymbol{p}}_n$ form an orthonormal basis of $\mathcal{R}^n$. It follows from (12) that

$$\boldsymbol{X}\check{\boldsymbol{p}}_i = 0 \text{ for every } (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}^* \text{ and every } i \in \{a+1, \cdots, n\}.$$

On the other hand, $\check{\boldsymbol{X}}\check{\boldsymbol{Y}} = \boldsymbol{O}$ since $(\check{\boldsymbol{X}}, \check{\boldsymbol{Y}}) \in \mathcal{F}^*$. Hence

$$0 = \check{\boldsymbol{p}}_j^\top \check{\boldsymbol{X}} \check{\boldsymbol{Y}} \check{\boldsymbol{p}}_j = \xi_j \check{\boldsymbol{p}}_j^\top \check{\boldsymbol{Y}} \check{\boldsymbol{p}}_j \text{ for every } j \in \{1, \cdots, a\}$$

which together with $\xi_j > 0 (j \in \{1, \cdots, a\})$ implies that

$$\check{\boldsymbol{p}}_j^\top \check{\boldsymbol{Y}} \check{\boldsymbol{p}}_j = 0 \text{ for every } j \in \{1, \cdots, a\},$$

or equivalently $\check{\boldsymbol{p}}_j \in V$ for every $j \in \{1, \cdots, a\}$. By (13), we then see that

$$\boldsymbol{Y}\check{\boldsymbol{p}}_j = 0 \text{ for every } (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}^* \text{ for every } j \in \{1, \cdots, a\}.$$

From this and the definition of the partition U, V and W, we can conclude $\mathcal{R}^n = U\dot{+}V\dot{+}W$. Replace $\{\check{\boldsymbol{p}}_{b+1}, \cdots, \check{\boldsymbol{p}}_n\}$ by linearly independent eigenvectors of $\check{\boldsymbol{Y}}$ corresponding to nonzero eigenvalues. Letting $J := \{1, \cdots, a\}$, $K := \{a+1, \cdots, b\}$, $I := \{b+1, \cdots, n\}$ and $(\hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}}) := (\check{\boldsymbol{X}}, \check{\boldsymbol{Y}})$, we obtain the desired result.

∎

4 Convergence of Trajectory of Semidefinite Linear Complementarity Problems.

In this section, we deal with the monotone semidefinite linear complementarity problem (SDLCP) *i.e.*, the case that $\mathcal{F}$ is affine;

$$\begin{array}{lll} \textbf{SDLCP} & \text{Find an} & (\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{S}_+ \times \mathcal{S}_+, \\ & \text{such that} & (\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{F} \text{ and } \boldsymbol{X} \bullet \boldsymbol{Y} = 0, \end{array} \tag{14}$$

where $\mathcal{F}$ is a maximal monotone affine subspace of $\mathcal{S} \times \mathcal{S}$. The purpose of this section is to show that the trajectory T^L with a linear continuation ($\boldsymbol{a}(t) = t\bar{\boldsymbol{a}}$, $\boldsymbol{B}(t) = t\bar{\boldsymbol{B}}$) converges to a specific solution of the monotone SDLCP under the *strict complementarity condition*, *i.e.*, there is a solution $(\tilde{\boldsymbol{X}},\tilde{\boldsymbol{Y}})$ of the SDLCP (14) such that $\tilde{\boldsymbol{X}} + \tilde{\boldsymbol{Y}} \succ \boldsymbol{O}$, or equivalently rank $\tilde{\boldsymbol{X}}$ +rank $\tilde{\boldsymbol{Y}} = n$ where $n = \sum n^q$, we call such a solution $(\tilde{\boldsymbol{X}},\tilde{\boldsymbol{Y}})$ a *strictly complementary solution*. Note that, as we mentioned in Section 2, the trajectory T of the monotone SDCP (1) does not necessarily converge to a single point, *i.e.*, there may be more than one accumulation point.

Theorem 4.1. *Let disjoint index subsets I^q, J^q and $K^q(=\emptyset)$, a solution $(\hat{\boldsymbol{X}},\hat{\boldsymbol{Y}}) \in \mathcal{F}^*$ and an orthogonal matrix $\boldsymbol{P}$ be as in Theorem 3.1. Let*

$$(\boldsymbol{X}(t),\boldsymbol{Y}(t)) = \boldsymbol{c}(t\bar{\boldsymbol{a}},t\bar{\boldsymbol{B}})$$

for $\bar{\boldsymbol{a}} > 0$ and $\bar{\boldsymbol{B}} = (\bar{\boldsymbol{B}}_X,\bar{\boldsymbol{B}}_Y) \in \mathcal{B}_{++}$. Assume that the trajectory $T^L = \{(\boldsymbol{X}(t),\boldsymbol{Y}(t)) : t \in (0,1]\}$ with the linear continuation is bounded. Let $(\bar{\boldsymbol{X}},\bar{\boldsymbol{Y}})$ be one of accumulation points of the trajectory T^L. Then;

(i) *The point $(\bar{\boldsymbol{X}},\bar{\boldsymbol{Y}})$ is a solution of the SDLCP (14) such that $[\boldsymbol{P}^\top\bar{\boldsymbol{X}}\boldsymbol{P}]_J \succ \boldsymbol{O}$ and $[\boldsymbol{P}^\top\bar{\boldsymbol{Y}}\boldsymbol{P}]_I \succ \boldsymbol{O}$. (That is, the accumulation points of the trajectory are solution of maximal rank.)*

(ii) *Suppose that there is a strictly complementary solution of the SDLCP (14). Then the trajectory T^L converges to a unique solution $(\boldsymbol{X}^*,\boldsymbol{Y}^*)$ of the following concave maximization problem (15) over the set $\mathcal{F}^*$ of the optimal solutions of the SDLCP (14):*

$$\begin{array}{ll} \textit{Maximize} & \displaystyle\sum_{q=1}^{l} \bar{a}^q \left(\log\det[(\boldsymbol{P}^q)^\top\boldsymbol{X}^q\boldsymbol{P}^q]_{J^q} + \log\det[(\boldsymbol{P}^q)^\top\boldsymbol{Y}^q\boldsymbol{P}^q]_{I^q}\right) - (\bar{\boldsymbol{B}}_X \bullet \boldsymbol{Y} + \boldsymbol{X} \bullet \bar{\boldsymbol{B}}_Y) \\ \textit{subject to} & (\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{F}^*. \end{array} \tag{15}$$

For the monotone LCP, Kojima Mizuno and Noma [22] showed the same kind of result without the strict complementarity condition. Our proof for the SDLCP is based on the proof of [22, Theorem 4.1]. For the SDP (7), Luo, Sturm and Zhang [32] recently showed that, under the strict complementarity condition, the feasible central trajectory $\{(\boldsymbol{X},\boldsymbol{Y}) \in$

$\mathcal{S}_+ \times \mathcal{S}_+ : (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}, \boldsymbol{XY} = \mu \boldsymbol{I}, \mu > 0\}$ converges to the unique solution $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$ of the optimal face such that

$$\left\{ \begin{array}{l} [\boldsymbol{P}^\top \boldsymbol{X}^* \boldsymbol{P}]_J \boldsymbol{V}_J = \boldsymbol{I}, \\ \boldsymbol{U}_I [\boldsymbol{P}^\top \boldsymbol{Y}^* \boldsymbol{P}]_I = \boldsymbol{I}, \\ \sum_{i=1}^m y_i [\boldsymbol{P}^\top \boldsymbol{A}_i \boldsymbol{P}]_J + \boldsymbol{V}_J = \boldsymbol{O}, \\ [\boldsymbol{P}^\top \boldsymbol{A}_i \boldsymbol{P}]_I \bullet \boldsymbol{U}_I = 0, \\ (\boldsymbol{X}^*, \boldsymbol{Y}^*) \in \mathcal{F}^* \text{ and } \boldsymbol{U}_I \succeq \boldsymbol{O}, \boldsymbol{V}_J \succeq \boldsymbol{O}. \end{array} \right. \tag{16}$$

Goldfarb and Scheinberg [11] independently showed for the SDP (7) that the feasible central trajectory converges to the analytic center $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$ of the optimal faces, *i.e.*, $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$ is a solution of the following concave maximization problem:

$$\begin{array}{ll} \text{maximize} & \log\det[\boldsymbol{P}^\top \boldsymbol{X} \boldsymbol{P}]_J + \log\det[\boldsymbol{P}^\top \boldsymbol{Y} \boldsymbol{P}]_I \\ \text{subject to} & (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}^*. \end{array} \tag{17}$$

Note that the system (16) is the Karush-Kuhn-Tucker condition of the concave maximization problem (17). Our research is also independent from both researches [11, 32]. We also refer a note of Ye [58] for the homogeneous self-dual cone.

The following simple lemma is necessary for the proof of Theorem 4.1.

Lemma 4.2. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be $n \times n$ positive definite symmetric matrices such that $\boldsymbol{XY} = \boldsymbol{I}$, where $\boldsymbol{I}$ denotes the $n \times n$ identity matrix. Then $\boldsymbol{X}_J = \boldsymbol{Y}_J^{-1} + *$ where $*$ is a $|J| \times |J|$ positive semidefinite matrix.* ∎

Now we are in a position to prove Theorem 4.1.

Proof of Theorem 4.1:
Let $\{t_k\}_{k=1,2,\cdots}$ be a positive sequence such that $\lim_{k\to\infty} (\boldsymbol{X}(t_k), \boldsymbol{Y}(t_k), t_k) = (\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}}, 0)$.
Part (i): By Theorem 2.2, the trajectory accumulates into the solution set of the SDLCP (14). For the sake of simplicity, we assume that $l = 1$. Then

$$\begin{array}{rcl} 0 & \leq & \hat{\boldsymbol{X}} \bullet \boldsymbol{Y}(t_k) + \hat{\boldsymbol{Y}} \bullet \boldsymbol{X}(t_k) \\ & \leq & -\left(t_k \bar{\boldsymbol{B}}_X \bullet (\boldsymbol{Y}(t_k) - \hat{\boldsymbol{Y}}) + t_k (\boldsymbol{X}(t_k) - \hat{\boldsymbol{X}}) \bullet \bar{\boldsymbol{B}}_Y \right) \\ & & \qquad + (t_k^2 \bar{\boldsymbol{B}}_X \bullet \bar{\boldsymbol{B}}_Y + \boldsymbol{X}(t_k) \bullet \boldsymbol{Y}(t_k) + \hat{\boldsymbol{X}} \bullet \hat{\boldsymbol{Y}} \text{ (by monotonicity)} \\ & = & t_k \bar{a} \mathbf{Tr}\ \boldsymbol{I} - t_k \left(\bar{\boldsymbol{B}}_X \bullet (\boldsymbol{Y}(t_k) - \hat{\boldsymbol{Y}}) + (\boldsymbol{X}(t_k) - \hat{\boldsymbol{X}}) \bullet \bar{\boldsymbol{B}}_Y \right) + t_k^2 \bar{\boldsymbol{B}}_X \bullet \bar{\boldsymbol{B}}_Y. \end{array}$$

Hence,

$$\begin{array}{rcl} 0 & \leq & [\boldsymbol{P}^\top \hat{\boldsymbol{X}} \boldsymbol{P}] \bullet [\boldsymbol{P}^\top \boldsymbol{Y}(t_k) \boldsymbol{P}] + [\boldsymbol{P}^\top \hat{\boldsymbol{Y}} \boldsymbol{P}] \bullet [\boldsymbol{P}^\top \boldsymbol{X}(t_k) \boldsymbol{P}] \\ & \leq & t_k \mathbf{Tr}\ \bar{a} \boldsymbol{I} - t_k \left(\bar{\boldsymbol{B}}_X \bullet (\boldsymbol{Y}(t_k) - \hat{\boldsymbol{Y}}) + (\boldsymbol{X}(t_k) - \hat{\boldsymbol{X}}) \bullet \bar{\boldsymbol{B}}_Y \right) + t_k^2 \bar{\boldsymbol{B}}_X \bullet \bar{\boldsymbol{B}}_Y. \end{array}$$

Using the property $\boldsymbol{X}(t_k) \boldsymbol{Y}(t_k) = t_k \bar{a} \boldsymbol{I}$ and Lemma 4.2,

$$[\boldsymbol{P}^\top \boldsymbol{Y}(t_k) \boldsymbol{P}]_J = t_k \bar{a} ([\boldsymbol{P}^\top \boldsymbol{X}(t_k) \boldsymbol{P}]_J)^{-1} + *_1 \text{ and } [\boldsymbol{P}^\top \boldsymbol{X}(t_k) \boldsymbol{P}]_I = t_k \bar{a} ([\boldsymbol{P}^\top \boldsymbol{Y}(t_k) \boldsymbol{P}]_I)^{-1} + *_2,$$

where $*_1$ and $*_2$ are positive semidefinite matrices of an appropriate size. Then we have

$$\begin{array}{rcl} 0 \leq [\boldsymbol{P}^\top \hat{\boldsymbol{X}} \boldsymbol{P}]_J \bullet \{ t_k \bar{a} ([\boldsymbol{P}^\top \boldsymbol{X}(t_k) \boldsymbol{P}]_J)^{-1} + *_1 \} + [\boldsymbol{P}^\top \hat{\boldsymbol{Y}} \boldsymbol{P}]_I \bullet \{ t_k \bar{a} ([\boldsymbol{P}^\top \boldsymbol{Y}(t_k) \boldsymbol{P}]_I)^{-1} + *_2 \} \\ \leq t_k \mathbf{Tr}\ \bar{a} \boldsymbol{I} - t_k \left(\bar{\boldsymbol{B}}_X \bullet (\boldsymbol{Y}(t_k) - \hat{\boldsymbol{Y}}) + (\boldsymbol{X}(t_k) - \hat{\boldsymbol{X}}) \bullet \bar{\boldsymbol{B}}_Y \right) + t_k^2 \bar{\boldsymbol{B}}_X \bullet \bar{\boldsymbol{B}}_Y, \end{array}$$

and thus

$$\begin{aligned} 0 &\leq \bar{a}[\boldsymbol{P}^\top \hat{\boldsymbol{X}} \boldsymbol{P}]_J \bullet ([\boldsymbol{P}^\top \boldsymbol{X}(t_k)\boldsymbol{P}]_J)^{-1} + \bar{a}[\boldsymbol{P}^\top \hat{\boldsymbol{Y}} \boldsymbol{P}]_I \bullet ([\boldsymbol{P}^\top \boldsymbol{Y}(t_k)\boldsymbol{P}]_I)^{-1} \\ &\leq \mathbf{Tr}\ \bar{a}\boldsymbol{I} - \left(\bar{\boldsymbol{B}}_X \bullet (\boldsymbol{Y}(t_k) - \hat{\boldsymbol{Y}}) + (\boldsymbol{X}(t_k) - \hat{\boldsymbol{X}}) \bullet \bar{\boldsymbol{B}}_Y\right) + t_k \bar{\boldsymbol{B}}_X \bullet \bar{\boldsymbol{B}}_Y. \end{aligned} \tag{18}$$

By taking the limit in the inequality (18), we have the inequalities

$$[\boldsymbol{P}^\top \bar{\boldsymbol{X}} \boldsymbol{P}]_J \succ \boldsymbol{O} \text{ and } [\boldsymbol{P}^\top \bar{\boldsymbol{Y}} \boldsymbol{P}]_I \succ \boldsymbol{O}.$$

Hence, we have the assertion (i). **(ii):** Since there is a strictly complementarity solution, we see

$$\begin{gathered} (\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}}) \in \mathcal{F}^*, \\ [\boldsymbol{P}^\top \boldsymbol{X}^* \boldsymbol{P}]_{ij} = [\boldsymbol{P}^\top \hat{\boldsymbol{X}} \boldsymbol{P}]_{ij} = [\boldsymbol{P}^\top \bar{\boldsymbol{X}} \boldsymbol{P}]_{ij} = 0 \quad (i \in I \text{ or } j \in I), \\ [\boldsymbol{P}^\top \boldsymbol{Y}^* \boldsymbol{P}]_{ij} = [\boldsymbol{P}^\top \hat{\boldsymbol{Y}} \boldsymbol{P}]_{ij} = [\boldsymbol{P}^\top \bar{\boldsymbol{Y}} \boldsymbol{P}]_{ij} = 0 \quad (i \in J \text{ or } j \in J). \end{gathered}$$

Hence, by taking the limit in the inequality (18) with substituting $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$ for $(\hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}})$, we have

$$\begin{aligned} 0 &\leq \bar{a}[\boldsymbol{P}^\top \boldsymbol{X}^* \boldsymbol{P}]_J \bullet ([\boldsymbol{P}^\top \bar{\boldsymbol{X}} \boldsymbol{P}]_J)^{-1} + \bar{a}[\boldsymbol{P}^\top \boldsymbol{Y}^* \boldsymbol{P}]_I \bullet ([\boldsymbol{P}^\top \bar{\boldsymbol{Y}} \boldsymbol{P}]_I)^{-1} \\ &\leq \mathbf{Tr}\ \bar{a}\boldsymbol{I} - \left(\bar{\boldsymbol{B}}_X \bullet (\bar{\boldsymbol{Y}} - \boldsymbol{Y}^*) + (\bar{\boldsymbol{X}} - \boldsymbol{X}^*) \bullet \bar{\boldsymbol{B}}_Y\right). \end{aligned} \tag{19}$$

Thus,

$$\begin{gathered} \bar{a}\mathbf{Tr}\ \left\{[\boldsymbol{P}^\top \boldsymbol{X}^* \boldsymbol{P}]_J ([\boldsymbol{P}^\top \bar{\boldsymbol{X}} \boldsymbol{P}]_J)^{-1} - \boldsymbol{I}_{|J|}\right\} + \bar{a}\mathbf{Tr}\ \left\{[\boldsymbol{P}^\top \boldsymbol{Y}^* \boldsymbol{P}]_I ([\boldsymbol{P}^\top \bar{\boldsymbol{Y}} \boldsymbol{P}]_I)^{-1} - \boldsymbol{I}_{|I|}\right\} \\ \leq \left(\bar{\boldsymbol{B}}_X \bullet (\boldsymbol{Y}^* - \bar{\boldsymbol{Y}}) + (\boldsymbol{X}^* - \bar{\boldsymbol{X}}) \bullet \bar{\boldsymbol{B}}_Y\right). \end{gathered} \tag{20}$$

(Remark that $I \cup J = \{1, \cdots, n\}$ by the strict complementarity assumption.) Let λ_j $(j \in J)$ and μ_i $(i \in I)$ denote the eigenvalues of the matrices

$$([\boldsymbol{P}^\top \bar{\boldsymbol{X}} \boldsymbol{P}]_J)^{-\frac{1}{2}} [\boldsymbol{P}^\top \boldsymbol{X}^* \boldsymbol{P}]_J ([\boldsymbol{P}^\top \bar{\boldsymbol{X}} \boldsymbol{P}]_J)^{-\frac{1}{2}} \text{ and } ([\boldsymbol{P}^\top \bar{\boldsymbol{Y}} \boldsymbol{P}]_I)^{-\frac{1}{2}} [\boldsymbol{P}^\top \boldsymbol{Y}^* \boldsymbol{P}]_I ([\boldsymbol{P}^\top \bar{\boldsymbol{Y}} \boldsymbol{P}]_I)^{-\frac{1}{2}},$$

respectively. Then, (20) is equivalent to the following inequality

$$\bar{a}\left\{\sum_{j\in J}(\lambda_j - 1) + \sum_{i\in I}(\mu_i - 1)\right\} \leq \left(\bar{\boldsymbol{B}}_X \bullet (\boldsymbol{Y}^* - \bar{\boldsymbol{Y}}) + (\boldsymbol{X}^* - \bar{\boldsymbol{X}}) \bullet \bar{\boldsymbol{B}}_Y\right).$$

Noting that $\log(z) \leq z - 1$ for every $z > 0$, we have

$$\bar{a}\left\{\sum_{j\in J}\log\lambda_j + \sum_{i\in I}\log\mu_i\right\} \leq \left(\bar{\boldsymbol{B}}_X \bullet (\boldsymbol{Y}^* - \bar{\boldsymbol{Y}}) + (\boldsymbol{X}^* - \bar{\boldsymbol{X}}) \bullet \bar{\boldsymbol{B}}_Y\right),$$

or equivalently,

$$\begin{gathered} \bar{a}\left\{\log\det[\boldsymbol{P}^\top \boldsymbol{X}^* \boldsymbol{P}]_J ([\boldsymbol{P}^\top \bar{\boldsymbol{X}} \boldsymbol{P}]_J)^{-1} + \log\det[\boldsymbol{P}^\top \boldsymbol{Y}^* \boldsymbol{P}]_I ([\boldsymbol{P}^\top \bar{\boldsymbol{Y}} \boldsymbol{P}]_I)^{-1}\right\} \\ \leq \left(\bar{\boldsymbol{B}}_X \bullet (\boldsymbol{Y}^* - \bar{\boldsymbol{Y}}) + (\boldsymbol{X}^* - \bar{\boldsymbol{X}}) \bullet \bar{\boldsymbol{B}}_Y\right). \end{gathered}$$

Thus, we obtain

$$\begin{aligned} &\bar{a}\left\{\log\det[\boldsymbol{P}^\top \boldsymbol{X}^* \boldsymbol{P}]_J + \log\det[\boldsymbol{P}^\top \boldsymbol{Y}^* \boldsymbol{P}]_I\right\} - \left(\bar{\boldsymbol{B}}_X \bullet \boldsymbol{Y}^* + \boldsymbol{X}^* \bullet \bar{\boldsymbol{B}}_Y\right) \\ &\leq \bar{a}\left\{\log\det[\boldsymbol{P}^\top \bar{\boldsymbol{X}} \boldsymbol{P}]_J + \log\det[\boldsymbol{P}^\top \bar{\boldsymbol{Y}} \boldsymbol{P}]_I\right\} - \left(\bar{\boldsymbol{B}}_X \bullet \bar{\boldsymbol{Y}} + \bar{\boldsymbol{X}} \bullet \bar{\boldsymbol{B}}_Y\right). \end{aligned}$$

Recall that $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$ is the unique maximum solution of the problem (15). Thus the last inequality ensures $(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}}) = (\boldsymbol{X}^*, \boldsymbol{Y}^*)$. Since the argument above holds for any accumulation point $(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}})$, the proof is complete. ∎

∎

5 A Generic Infeasible-Interior-Point Method.

In this section, we describe a generic infeasible-interior-point method to solve the SDCP (1) which globally converges into the solution set.

As we see in Section 3, the solution set $\mathcal{F}^*$ is nonempty, convex and compact if the interior $\mathcal{F}_{++}$ of the feasible set $\mathcal{F}_+$ is nonempty [20]. But, even if $\mathcal{F}$ is affine, degenerate case [$\mathcal{F}_+ \neq \emptyset,\ \mathcal{F}_{++} = \emptyset$] does not imply $\mathcal{F}^* \neq \emptyset$, see also [12, 39].

The SDCP (1), even if the manifold $\mathcal{F}$ is affine, has other difficulty to compute an exact solution. Even if all the input data are rational in the SDCP (1), the solution set $\mathcal{F}^*$ may have an irrational element. For example;

Example 5.1. Consider the following primal-dual pair of an SDP:

$$
\begin{array}{lll}
(\mathbf{P}): \text{Minimize} & \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \bullet \boldsymbol{X} & (\mathbf{D}): \text{Maximize} \quad 4z \\
\text{subject to} & \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \bullet \boldsymbol{X} = 4 & \text{subject to} \quad z \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \boldsymbol{Y} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \\
& \boldsymbol{X} \succeq \boldsymbol{O}. & \boldsymbol{Y} \succeq \boldsymbol{O}.
\end{array}
$$

Note that both problems have strictly feasible solutions ($\boldsymbol{X} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \succ \boldsymbol{O}$ is strictly feasible for (P), and $\boldsymbol{Y} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \succ \boldsymbol{O}, z = -1$ is strictly feasible for (D)). Hence, both problems (P) and (D) have optimal solutions and the duality gap ($\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \bullet \boldsymbol{X} - 4z$) is equal to 0 at the primal-dual optimal solutions ([1, 12]). It is easy to see that $\boldsymbol{X}^* = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$ and $z^* = -\dfrac{\sqrt{2}}{2}$, $\boldsymbol{Y}^* = \begin{pmatrix} \dfrac{\sqrt{2}}{2} & -1 \\ -1 & \sqrt{2} \end{pmatrix}$ are the unique optima for (P) and (D), respectively. Therefore,

$$
(\boldsymbol{X}^*, \boldsymbol{Y}^*) = \left(\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}, \begin{pmatrix} \dfrac{\sqrt{2}}{2} & -1 \\ -1 & \sqrt{2} \end{pmatrix} \right)
$$

is the unique solution, which have irrational elements, of the corresponding (monotone semidefinite) complementarity problem:

$$
\begin{array}{lll}
\textbf{SDLCP} & \text{Find} & (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}(2)_+ \times \mathcal{S}(2)_+ \\
& \text{such that} & \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \bullet \boldsymbol{X} = 4,\ z \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \boldsymbol{Y} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},\ z \in \mathcal{R} \\
& & \boldsymbol{X} \bullet \boldsymbol{Y} = 0.
\end{array}
$$

∎

Thus, a reasonable practical requirement for a numerical algorithm for the SDCP (1) seems to be:

(a) Given an $\epsilon > 0$ and a bounded subset C of $\mathcal{S} \times \mathcal{S}$, either compute an ϵ-approximate solution, *i.e.*, an $(\boldsymbol{X}, \boldsymbol{Y})$ satisfying

$$d((\boldsymbol{X}, \boldsymbol{Y}), \mathcal{F}) \leq \epsilon,\ (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_+ \times \mathcal{S}_+ \text{ and } \boldsymbol{X} \bullet \boldsymbol{Y} \leq \epsilon,$$

or detect that the SDCP (1) has no solution in the set C, where

$$d((\boldsymbol{X}, \boldsymbol{Y}), \mathcal{F}) := \inf\{\|(\boldsymbol{X}, \boldsymbol{Y}) - (\boldsymbol{X}', \boldsymbol{Y}')\| : (\boldsymbol{X}', \boldsymbol{Y}') \in \mathcal{F}\}$$

with an appropriate matrix norm $\|(\cdot, \cdot)\|$.

Alizadeh [1] presents a feasible-interior-point algorithm which approximately solve the SDP (7) in polynomial time if an *a priori* bound of the size of the solution is known. Note that the polynomial time solvability of the SDP (7) is a direct consequence of the general results based on the ellipsoid method for convex programming in 1979 (a note of Khachiyan, see [13]). Kojima, et al. [29] proposed polynomial time feasible/infeasible-interior-point algorithms for the monotone semidefinite linear complementarity problems (SDLCPs) under the assumption that the solution exists in a given appropriate bounded set. Freund [10], Monteiro [42] and Zhang [59] proposed the polynomial time feasible/infeasible-interior-point algorithm for the SDP. Recently, Potra and Sheng [48] and Kojima et al. [26, 28] investigated the global and local convergence of infeasible-interior-point algorithms for the SDP and the monotone SDLCP. Especially, Kojima et al. proposed the superlinear convergent infeasible-interior-point algorithms using different search directions originally defined in [2] and [29]. Luo et al. [32] proposed the superlinear convergent feasible-interior-point algorithms for the SDP using the search direction which is originally defined by Nesterov and Todd [45, 46] then independently derived by [54].

In this section, as in [25], we present a wide class of globally convergent infeasible-interior-point algorithms for the monotone semidefinite complementarity problems (SDCPs) (1) in terms of a unified global convergence theory given by Polak [47] in 1971 for nonlinear programs. In addition to (a), our interior-point algorithms are characterized as:

(b) Move in a search direction for approximating a point on the trajectory at each iteration.

(c) Start from an infeasible but positive definite initial point and generate a sequence of infeasible points.

A primal-dual infeasible-interior-point algorithm for linear programs was proposed by Lustig [33], and effectively implemented on the OB1 [38]. The approach was also developed for monotone nonlinear complementarity problems (CPs) in [25]. The algorithms for the SDP and the monotone semidefinite linear complementarity problems (SDLCPs) were proposed in [26, 28, 29, 48, 59].

To describe a generic infeasible-interior-point method (see [25]), we introduce

- an *admissible set* Ω in which we confine all iterates; $\Omega \subseteq (\mathcal{S}_{++} \times \mathcal{S}_{++}) \cup \mathcal{F}^*$ prevents an iterate from approaching the boundary of $\mathcal{S}_{++} \times \mathcal{S}_{++}$ before it approaches to the solution set $\mathcal{F}^*$ of the SDCP (1),

- a *merit function* ψ which gauges the progress of the iteration and is used to determine a step length at each iteration,
- an *algorithm mapping* $\mathcal{A}_{lg}$ which gives us a next iterate.

We impose some assumptions on Ω, ψ and $\mathcal{A}_{lg}$ below. Condition 5.2 ensures the global convergence and detects whether the SDCP (1) has no solution in a given bounded subset of $\mathcal{S}_+ \times \mathcal{S}_+$.

Condition 5.2.

1. $(\boldsymbol{X}_0, \boldsymbol{Y}_0) \in \Omega$, $(\bar{\boldsymbol{X}}_0, \bar{\boldsymbol{Y}}_0) \in \mathcal{F}$ are known. Put $\boldsymbol{B} = (\boldsymbol{B}_X, \boldsymbol{B}_Y) = (\boldsymbol{X}_0, \boldsymbol{Y}_0) - (\bar{\boldsymbol{X}}_0, \bar{\boldsymbol{Y}}_0)$.
2. $\Omega = \Omega_{++} \cup \Omega^*$ is a subset of $(\mathcal{S}_{++} \times \mathcal{S}_{++}) \cup \mathcal{F}^*$ and closed in $\mathcal{S} \times \mathcal{S}$, where
$$\Omega_{++} \subset \{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_{++} \times \mathcal{S}_{++} : (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F} + \theta \boldsymbol{B} \text{ for some } \theta \in [0, 1]\}$$
and $\Omega^* \subset \mathcal{F}^*$.
3. ψ is a real valued continuous function on Ω_{++} such that
$$\sup\{\boldsymbol{X} \bullet \boldsymbol{Y} : \ (\boldsymbol{X}, \boldsymbol{Y}) \in \Omega_{++}, \ \psi(\boldsymbol{X}, \boldsymbol{Y}) \leq t\} < \infty$$
for every sufficiently large $t \in \mathcal{R}$.
4. $\mathcal{A}_{lg}$ is a point-to-set mapping from Ω_{++} into the collection of nonempty subsets of Ω. For every $(\boldsymbol{X}', \boldsymbol{Y}') \in \Omega_{++}$, there exist positive numbers δ and ϵ such that if $(\boldsymbol{X}, \boldsymbol{Y}) \in D((\boldsymbol{X}', \boldsymbol{Y}'), \delta) \cap \Omega$ and $(\boldsymbol{X}^*, \boldsymbol{Y}^*) \in \mathcal{A}_{lg}(\boldsymbol{X}, \boldsymbol{Y})$ then either
$$(\boldsymbol{X}^*, \boldsymbol{Y}^*) \in \Omega^* \ \text{ or } \ (\boldsymbol{X}^*, \boldsymbol{Y}^*) \in \Omega_{++} \ \text{ and } \psi(\boldsymbol{X}^*, \boldsymbol{Y}^*) \leq \psi(\boldsymbol{X}, \boldsymbol{Y}) - \epsilon.$$
holds, where $D((\boldsymbol{X}', \boldsymbol{Y}'), \delta) := \{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S} \times \mathcal{S} : \|(\boldsymbol{X}, \boldsymbol{Y}) - (\boldsymbol{X}', \boldsymbol{Y}')\| \leq \delta\}$ with an appropriate matrix norm $\|(\cdot, \cdot)\|$.
5. if $(\boldsymbol{X}, \boldsymbol{Y}) \in \Omega_{++} \cap (\mathcal{F} + \theta \boldsymbol{B})$ for some $\theta \in [0, 1]$ and $(\boldsymbol{X}^*, \boldsymbol{Y}^*) \in \mathcal{A}_{lg}(\boldsymbol{X}, \boldsymbol{Y})$ then
$$(\boldsymbol{X}^*, \boldsymbol{Y}^*) \in \mathcal{F} + (1 - \alpha)\theta \boldsymbol{B} \text{ for some } \alpha \in [0, 1],$$
where $\alpha \in [0, 1]$ corresponds to a step length. ∎

We put off detailed discussions about how to construct an algorithm mapping $\mathcal{A}_{lg}$ satisfying Condition 5.2.

Generic IIP Method.

Step 0: Let $k = 0$. Let C be a given bounded subset of $\mathcal{S}_+ \times \mathcal{S}_+$

Step 1: If $(\boldsymbol{X}_k, \boldsymbol{Y}_k) \in \Omega^*$ then stop.

Step 2: If we detect that $\mathcal{F}^* \cap C = \emptyset$ (for example,Lemma 5.3 (C') below), then stop.

Step 3: Choose an $(\boldsymbol{X}_{k+1}, \boldsymbol{Y}_{k+1}) \in \mathcal{A}_{lg}(\boldsymbol{X}_k, \boldsymbol{Y}_k)$. Replace k by $k + 1$ and go to Step 1.

Let $\{(\boldsymbol{X}_k, \boldsymbol{Y}_k)\}$ be the sequence generated by the Generic IIP Method. By Condition 5.2, we see that for every $k = 0, 1, \cdots$,

$$(\boldsymbol{X}_k, \boldsymbol{Y}_k) \in \Omega_{++} \text{ and } \psi(\boldsymbol{X}_k, \boldsymbol{Y}_k) > \psi(\boldsymbol{X}_{k+1}, \boldsymbol{Y}_{k+1}) \text{ if } (\boldsymbol{X}_{k+1}, \boldsymbol{Y}_{k+1}) \notin \mathcal{F}^*.$$

Thus, the Generic IIP Method may be regarded as a feasible descent method for the artificial nonlinear programming problem:

$$\begin{array}{ll} \text{Minimize} & \psi(\boldsymbol{X}, \boldsymbol{Y}) \\ \text{subject to} & (\boldsymbol{X}, \boldsymbol{Y}) \in \Omega. \end{array}$$

Remark that, by the construction of the generated sequence under Condition 5.2,

$$\begin{array}{l} (\bar{\boldsymbol{X}}_k, \bar{\boldsymbol{Y}}_k) \equiv (\boldsymbol{X}_k, \boldsymbol{Y}_k) - \theta_k \boldsymbol{B} \in \mathcal{F}, \\ 1 = \theta_0 \geq \theta_k \geq \theta_{k+1} \geq 0 \end{array}$$

for all $k = 1, 2, \cdots$.

Lemma 5.3. *Let ϵ be a sufficiently small positive number and M be a sufficiently large positive number. Then there exists a finite number k such that at least one of the following (A)', (B)' and (C)' occurs.*

(A)' $\boldsymbol{X}_k \bullet \boldsymbol{Y}_k \leq \epsilon$ *and* $\theta_k \leq \epsilon$.

(B)' $\boldsymbol{B} \neq (\boldsymbol{O}, \boldsymbol{O})$ *and* $\theta_k \leq \epsilon$.

(C)' $\boldsymbol{B} \neq (\boldsymbol{O}, \boldsymbol{O})$, $\theta_k \geq \epsilon$ *and*

$$-\boldsymbol{X}_k \bullet \boldsymbol{Y}_k + \theta_k(\boldsymbol{X}_k \bullet \boldsymbol{B}_Y + \boldsymbol{B}_X \bullet \boldsymbol{Y}_k) - (\theta_k)^2 \boldsymbol{B}_X \bullet \boldsymbol{B}_Y \geq \theta_k M.$$

In the case (C)', the set

$$C(M) = \{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}_+ : (\boldsymbol{X} \bullet \boldsymbol{B}_Y + \boldsymbol{B}_X \bullet \boldsymbol{Y}) < M\}$$

contains no solution of the SDCP (1).

Proof: See Appendix. ∎

Lemma 5.3 ensures that if a feasible interior point $(\boldsymbol{X}_0, \boldsymbol{Y}_0) \in \mathcal{F}_{++}$ with the residual $\boldsymbol{B} = \boldsymbol{O}$ is available then the case (A)' occurs. In the case (A)', we have an approximate solution $(\boldsymbol{X}_p, \boldsymbol{Y}_p)$ of the SDCP (1) such that $\theta_p \leq \epsilon$, $(\boldsymbol{X}_p, \boldsymbol{Y}_p) \succeq \boldsymbol{O}$ and $\boldsymbol{X}_p \bullet \boldsymbol{Y}_p \leq \epsilon$. Hence we stop the iteration. In the case (B)', the point $(\boldsymbol{X}_p, \boldsymbol{Y}_p)$ may be regarded as an approximate feasible solution of the SDCP (1) although its total complementarity $\boldsymbol{X}_p \bullet \boldsymbol{Y}_p$ may not attain the desired accuracy $\boldsymbol{X} \bullet \boldsymbol{Y} \leq \epsilon$. More precisely, $(\boldsymbol{X}_p, \boldsymbol{Y}_p)$ is a feasible interior solution of the following complementarity problem with the fixed perturbation term $\theta_p \boldsymbol{B}$ satisfying $\theta_p \leq \epsilon$:

$$\text{Find } (\boldsymbol{X}, \boldsymbol{Y}) \text{ such that } (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}_+(\theta_p \boldsymbol{B}) \text{ and } \boldsymbol{X} \bullet \boldsymbol{Y} = 0. \tag{21}$$

Let $\{(\boldsymbol{X}_k, \boldsymbol{Y}_k) : k = p, p+1, \cdots\}$ be a sequence generated by the Generic IIP Method to the perturbed problem (21) starting from the feasible interior solution $(\boldsymbol{X}_p, \boldsymbol{Y}_p)$. Then the initial residual vector $\boldsymbol{B}$ is $\boldsymbol{O}$; hence, by Lemma 5.3, there is a finite number $q \geq p$ for which $(\boldsymbol{X}_q, \boldsymbol{Y}_q) \in \mathcal{F}_+(\theta_p \boldsymbol{B})$ and $\boldsymbol{X}_q \bullet \boldsymbol{Y}_q \leq \epsilon$. Thus $(\boldsymbol{X}_q, \boldsymbol{Y}_q)$ gives an approximate solution of the original SDCP (1).

It should be noted that any given bounded subset of $\mathcal{S}_+ \times \mathcal{S}_+$ is contained by the set $C(M)$ if we take M sufficiently large. Hence if the SDCP (1) has a solution, either the case (A)' or (B)' must occur for every sufficiently large M.

Ingredients.

In the rest of this section, we give ingredient of the Generic IIP Method. We assume that the maximal monotone set $\mathcal{F}$ is a C^1-manifold in $\mathcal{S} \times \mathcal{S}$.

5.1 Admissible Set and Merit Function.

Let $0 \leq \underline{\beta} < 1$, $0 \leq \overline{\gamma} < 1$ and $\nu : \mathcal{R}_+ \to \mathcal{R}_+$ be a C^1-function such that

$$\left.\begin{array}{l} 0 = \nu(0) < \nu(t) \ \text{ and} \\ 0 < (1-\overline{\gamma})\nu(t) - (1-\underline{\beta})\nu'(t)t, \\ \nu'(t) \geq 0 \end{array}\right\} \text{ for every } t > 0. \tag{22}$$

For example, $\nu(t) := ct^{\frac{1-\overline{\gamma}-\epsilon}{1-\underline{\beta}}}$ with a positive constant c and a positive ϵ such that $\overline{\gamma} + \epsilon < 1$. Especially, if $\underline{\beta} > \overline{\gamma}$, we can employ a linear function ν of the form $\nu(t) = ct$ with a positive constant c. If $\underline{\beta} \geq \overline{\gamma}$, we can take a strictly increasing concave C^2-function $\nu : \mathcal{R}_+ \to \mathcal{R}_+$ such that

$$0 = \nu(0), \ \nu'(t) > 0 \ \text{ and } \nu''(t) < 0 \ \text{ for every } t > 0;$$

for example, $\nu(t) = \log(1+t)$ for every $t \geq 0$.

Let

$$\begin{array}{rcl} (\boldsymbol{X}_0, \boldsymbol{Y}_0) & \succ & \boldsymbol{O}, \\ (\bar{\boldsymbol{X}}_0, \bar{\boldsymbol{Y}}_0) & \in & \mathcal{F} \\ 1 & \leq & \nu(\boldsymbol{X}_0 \bullet \boldsymbol{Y}_0), \\ \psi(\boldsymbol{X}, \boldsymbol{Y}) & = & \rho \log \boldsymbol{X} \bullet \boldsymbol{Y} - \log\det \boldsymbol{X}\boldsymbol{Y} \text{ for some } \rho > n, \\ \Omega_{++} & = & \left\{ (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S} \times \mathcal{S} : \begin{array}{c} (\boldsymbol{X}, \boldsymbol{Y}) \succ \boldsymbol{O}, \ \psi(\boldsymbol{X}, \boldsymbol{Y}) \leq \psi(\boldsymbol{X}_0, \boldsymbol{Y}_0), \\ (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F} + \theta\left((\boldsymbol{X}_0, \boldsymbol{Y}_0) - (\bar{\boldsymbol{X}}_0, \bar{\boldsymbol{Y}}_0)\right) \\ \text{and } \theta \leq \nu(\boldsymbol{X} \bullet \boldsymbol{Y}) \ \text{ for some } \theta \in [0,1] \end{array} \right\}, \\ \Omega^* & = & \mathcal{F}^*, \\ \Omega & = & \Omega_{++} \cup \Omega^*. \end{array}$$

The inequality $\psi(\boldsymbol{X}, \boldsymbol{Y}) \leq \psi(\boldsymbol{X}_0, \boldsymbol{Y}_0)$ in the definition of Ω_{++} prevents the iterate $(\boldsymbol{X}_k, \boldsymbol{Y}_k)$ from approaching to the boundary of $\mathcal{S}_{++} \times \mathcal{S}_{++}$ before it approaches to the solution set $\mathcal{F}^*$ of the SDCP.

Remark 5.4. We can use some different types of a pair (Ω, ψ) as we can see in [25, Section 4] with a small modification. ∎

5.2 Search Directions.

Assume that

$$(\boldsymbol{X},\boldsymbol{Y}) = (\boldsymbol{X}_k,\boldsymbol{Y}_k) \in \mathcal{S}_{++}\times\mathcal{S}_{++},$$
$$(\bar{\boldsymbol{X}},\bar{\boldsymbol{Y}}) = (\bar{\boldsymbol{X}}_k,\bar{\boldsymbol{Y}}_k) = (\boldsymbol{X}_k,\boldsymbol{Y}_k) - \theta_k\boldsymbol{B} \in \mathcal{F},$$
$$0 \le \theta_k \le 1.$$

Let $(\boldsymbol{X}_\gamma,\boldsymbol{Y}_\gamma) := \gamma(\boldsymbol{X},\boldsymbol{Y})+(1-\gamma)(\bar{\boldsymbol{X}},\bar{\boldsymbol{Y}})$ for each $\gamma\in[0,1]$. Let $\mathcal{T}_k(\gamma)$ denote the tangent plane of $\mathcal{F}(\gamma\theta_k\boldsymbol{B})$ at $(\boldsymbol{X}_\gamma,\boldsymbol{Y}_\gamma)$. Note that

$$\mathcal{T}_k(\gamma) = (\boldsymbol{X}_\gamma,\boldsymbol{Y}_\gamma) + \mathcal{L}_k,$$

where $\mathcal{L}_k$ is the tangent space of $\mathcal{F}$ at $(\bar{\boldsymbol{X}}_k,\bar{\boldsymbol{Y}}_k)$ (and also the tangent space of $\mathcal{F}(\gamma\theta_k\boldsymbol{B})$ at $(\boldsymbol{X}_\gamma,\boldsymbol{Y}_\gamma)$).

We consider the target point as follows

$$\{(\boldsymbol{X},\boldsymbol{Y})\in\mathcal{S}_+\times\mathcal{S}_+ : \boldsymbol{X}\boldsymbol{Y} = \beta\mu_k\boldsymbol{I}, (\boldsymbol{X},\boldsymbol{Y})\in\mathcal{F}(\gamma\theta_k\boldsymbol{B})\},$$

i.e., $\boldsymbol{c}(\beta\mu_k\boldsymbol{e},\gamma\theta_k\boldsymbol{B})$, where $\beta\in[\underline{\beta},1)$, $\gamma\in[0,\overline{\gamma}]$ are parameters, $\mu_k := \boldsymbol{X}_k\bullet\boldsymbol{Y}_k/n$ ($n=\sum_{q=1}^{l}n^q$) and $\boldsymbol{e}:=(1,\cdots,1)^\top\in\mathcal{R}^n$. For example, for all k,

if $\beta=\gamma$, the target point lies on the trajectory with a linear continuation,
if $\gamma=0$, the target point lies on the feasible trajectory, *i.e.*, $\boldsymbol{c}(\beta\mu_k\boldsymbol{e},\boldsymbol{O})$.

To get the symmetric search direction $(d\boldsymbol{X},d\boldsymbol{Y})\in\mathcal{S}\times\mathcal{S}$, there are some systems which approximate the trajectory $\{(\boldsymbol{X},\boldsymbol{Y})\in\mathcal{F}\cap(\mathcal{S}_+\times\mathcal{S}_+) : \boldsymbol{X}\boldsymbol{Y}=a\boldsymbol{I}\}$. For example, we choose as the search direction $(d\boldsymbol{X},d\boldsymbol{Y})\in\mathcal{S}\times\mathcal{S}$ the intersection of

$$\{(d\boldsymbol{X},d\boldsymbol{Y}) : (\boldsymbol{X}_k+d\boldsymbol{X}-\boldsymbol{X}_\gamma,\boldsymbol{Y}_k+d\boldsymbol{Y}-\boldsymbol{Y}_\gamma)\in\mathcal{L}_k\}$$
and $\{(d\boldsymbol{X},d\boldsymbol{Y}) :$ the solution of the system **(i)**$\}$,

where $\mathbf{i}=\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$ or $\mathbf{3}$.

(0): $\boldsymbol{X}_k d\boldsymbol{Y}+d\boldsymbol{X}\boldsymbol{Y}_k+\boldsymbol{Y}_k d\boldsymbol{X}+d\boldsymbol{Y}\boldsymbol{X}_k = 2\beta\mu\boldsymbol{I}-(\boldsymbol{X}_k\boldsymbol{Y}_k+\boldsymbol{Y}_k\boldsymbol{X}_k)$ [1] (the sum of the linear approximations $\boldsymbol{X}\boldsymbol{Y}=\beta\mu\boldsymbol{I}$ and $\boldsymbol{Y}\boldsymbol{X}=\beta\mu\boldsymbol{I}$). The intersection with $\mathcal{F}$ might be empty (though, in generic, it is a singleton), see [26, 51]. However, the computational behavior using the search direction obtained by the system is stable. Recently, Kojima et al. [28] showed the polynomial time and superlinear convergence of the predictor-corrector infeasible-interior-point algorithm for the SDP and the monotone SDLCP.

(1): $\boldsymbol{X}_k(d\boldsymbol{Y}+\widetilde{d\boldsymbol{Y}})+(d\boldsymbol{X}+\widetilde{d\boldsymbol{X}})\boldsymbol{Y}_k = \beta\mu\boldsymbol{I}-\boldsymbol{X}_k\boldsymbol{Y}_k$ [29], where $(\widetilde{d\boldsymbol{X}},\widetilde{d\boldsymbol{Y}})$ is in a maximal monotone subspace $\mathcal{F}'$ of $\mathcal{S}^\perp$, where $\mathcal{S}^\perp$ denotes the class of $(n\times n)$-skew-symmetric matrices, (the approximation of $\boldsymbol{X}\boldsymbol{Y}=\beta\mu\boldsymbol{I}$). Kojima et al. [26] established the polynomial time and superlinear convergence of a predictor-corrector infeasible-interior-point algorithm for the SDP and the monotone SDLCP.

(2): $\boldsymbol{X}_k^{-\frac{1}{2}}(d\boldsymbol{X}\boldsymbol{Y}_k+\boldsymbol{X}_k d\boldsymbol{Y})\boldsymbol{X}_k^{\frac{1}{2}}+\boldsymbol{X}^{\frac{1}{2}}(d\boldsymbol{Y}\boldsymbol{X}_k+\boldsymbol{Y}_k d\boldsymbol{X})\boldsymbol{X}_k^{-\frac{1}{2}} = 2(\beta\mu\boldsymbol{I}-\boldsymbol{X}_k^{\frac{1}{2}}\boldsymbol{Y}_k\boldsymbol{X}_k^{\frac{1}{2}})$ [42], (the scaled sum of the linear approximation of $\boldsymbol{X}\boldsymbol{Y}=\beta\mu\boldsymbol{I}$ and $\boldsymbol{Y}\boldsymbol{X}=\beta\mu\boldsymbol{I}$). Kojima and Monteiro pointed out that the search direction is equal to the search direction obtained by **(1)** by taking $\mathcal{F}'=\{(\boldsymbol{O},\hat{\boldsymbol{V}}) : \hat{\boldsymbol{V}}\in\mathcal{S}^\perp\}$. Monteiro [42] and Zhang

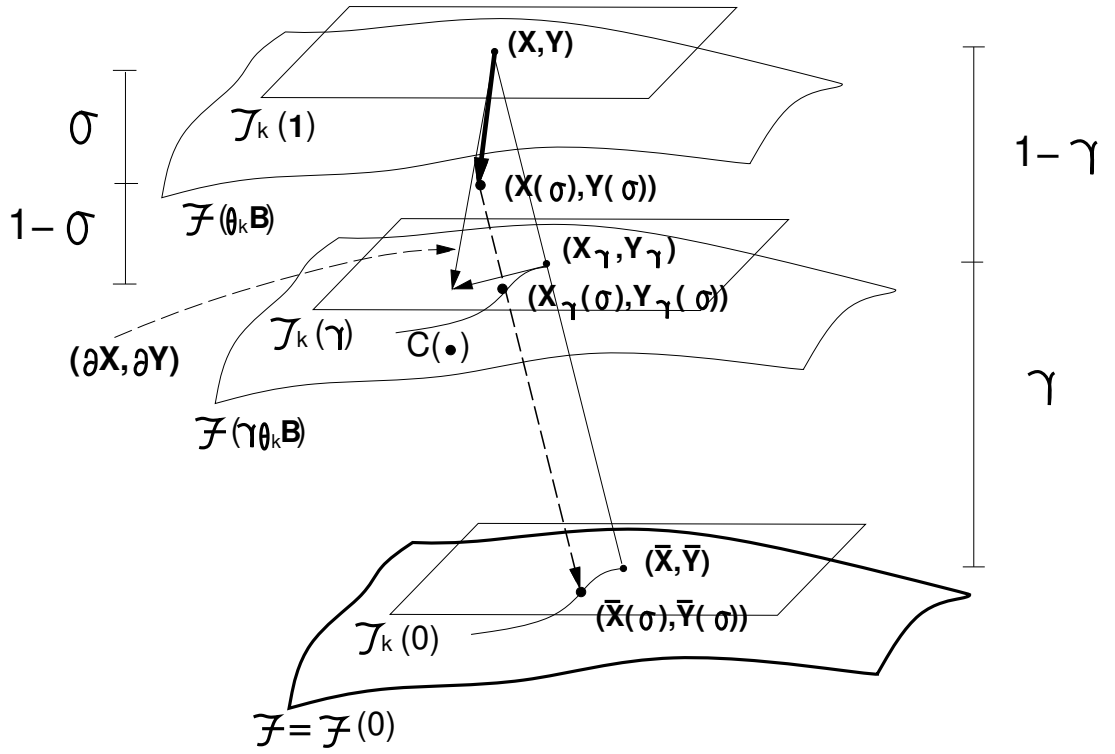

Table 2:

[59] proposed a polynomial time predictor-corrector feasible/infeasible-interior-point algorithms for the SDP. See also Potra and Sheng [48] for the local convergence of the SDP.

(3): $\boldsymbol{S}_k^{-1} d\boldsymbol{X} \boldsymbol{S}_k^{-1} + d\boldsymbol{Y} = \beta\mu_k \boldsymbol{X}_k^{-1} - \boldsymbol{Y}_k$ [45, 46], where $\boldsymbol{S}_k = \boldsymbol{X}_k^{\frac{1}{2}} (\boldsymbol{X}_k^{\frac{1}{2}} \boldsymbol{Y}_k \boldsymbol{X}_k^{\frac{1}{2}})^{-\frac{1}{2}} \boldsymbol{X}_k^{\frac{1}{2}}$ is a scaling point. Recently, Sturm and Zhang [54] independently derived this search direction and proposed the polynomial time predictor-corrector feasible-interior-point algorithm for the SDP using the direction. Luo et al. [32] showed the superlinear convergence of the modified algorithm for the SDP.

In generic, the search direction is uniquely determined since the number of the variables consists with the number of the equations. Moreover, using the systems **(1)**, **(2)** and **(3)**, the search direction is a singleton. For more detail, see [51].

5.3 Algorithm Mapping.

By the definition of tangent plane $\mathcal{T}_k(\gamma)$, there exists a C^1-curve $C(t)$ on $\mathcal{F}(\gamma\theta_k \boldsymbol{B})$ such that

$$C(0) = (\boldsymbol{X}_\gamma, \boldsymbol{Y}_\gamma) \text{ and } \dot{C}(0) = (\boldsymbol{X}_k + d\boldsymbol{X}, \boldsymbol{Y}_k + d\boldsymbol{Y}) - (\boldsymbol{X}_\gamma, \boldsymbol{Y}_\gamma) \in \mathcal{L}_k.$$

Define

$$\begin{aligned}
(\boldsymbol{X}_\gamma(\sigma), \boldsymbol{Y}_\gamma(\sigma)) &= C(\sigma) \in \mathcal{F}(\gamma\theta_k \boldsymbol{B}), \\
(\bar{\boldsymbol{X}}(\sigma), \bar{\boldsymbol{Y}}(\sigma)) &= C(\sigma) - \gamma\theta_k \boldsymbol{B} \in \mathcal{F}, \\
(\boldsymbol{X}(\sigma), \boldsymbol{Y}(\sigma)) &= (1-\sigma)\left((\boldsymbol{X}_k, \boldsymbol{Y}_k) - (\boldsymbol{X}_\gamma, \boldsymbol{Y}_\gamma)\right) + C(\sigma) \\
&= (1-\sigma)(1-\gamma)\theta_k \boldsymbol{B} + (\boldsymbol{X}_\gamma(\sigma), \boldsymbol{Y}_\gamma(\sigma)) \in \mathcal{F}\left((1-(1-\gamma)\sigma)\theta_k \boldsymbol{B}\right).
\end{aligned} \tag{23}$$

Then

$$\begin{aligned}
(\boldsymbol{X}(\sigma), \boldsymbol{Y}(\sigma)) &= (1-\sigma)\left((\boldsymbol{X}_k, \boldsymbol{Y}_k) - (\boldsymbol{X}_\gamma, \boldsymbol{Y}_\gamma)\right) + C(\sigma) \\
&= (1-\sigma)\left((\boldsymbol{X}_k, \boldsymbol{Y}_k) - (\boldsymbol{X}_\gamma, \boldsymbol{Y}_\gamma)\right) + (\boldsymbol{X}_\gamma, \boldsymbol{Y}_\gamma) + \sigma \dot{C}(0) + \boldsymbol{o}(\sigma) \\
&= (\boldsymbol{X}_k, \boldsymbol{Y}_k) + \sigma(d\boldsymbol{X}, d\boldsymbol{Y}) + \boldsymbol{o}(\sigma).
\end{aligned}$$

We have to show that $(\boldsymbol{X}(\sigma), \boldsymbol{Y}(\sigma)) \in \Omega$ for any sufficiently small positive σ.

Theorem 5.5. *Suppose that $(\boldsymbol{X}, \boldsymbol{Y}) \in \Omega_{++}$. Then $(\boldsymbol{X}(\sigma), \boldsymbol{Y}(\sigma)) \in \Omega$ for any sufficiently small positive σ.*

Proof: To show this, we only have to prove that

$$\psi(\boldsymbol{X}(\sigma), \boldsymbol{Y}(\sigma)) \le \psi(\boldsymbol{X}, \boldsymbol{Y}), \tag{24}$$

$$\text{and if} \quad \nu(\boldsymbol{X} \bullet \boldsymbol{Y}) \ge \theta \text{ then } \nu(\boldsymbol{X}(\sigma) \bullet \boldsymbol{Y}(\sigma)) \ge (1-(1-\gamma)\sigma)\,\theta. \tag{25}$$

We can prove the inequality (24) as [25, Lemma 4.4]. In the following, we shall continue to prove the inequality (25). Since ν is a C^1 function with properties (22) in Subsection 5.1,

$$\begin{aligned}
\frac{\partial \nu(\boldsymbol{X}(\sigma) \bullet \boldsymbol{Y}(\sigma))}{\partial \sigma}\Big|_{\sigma=0} &= \nu'(\boldsymbol{X} \bullet \boldsymbol{Y}) \frac{\partial (\boldsymbol{X}(\sigma) \bullet \boldsymbol{Y}(\sigma))}{\partial \sigma}\Big|_{\sigma=0} \\
&= -\nu'(\boldsymbol{X} \bullet \boldsymbol{Y})\,(1-\beta)\boldsymbol{X} \bullet \boldsymbol{Y} \\
&\ge -\nu'(\boldsymbol{X} \bullet \boldsymbol{Y})\,(1-\underline{\beta})\boldsymbol{X} \bullet \boldsymbol{Y} \\
&> -(1-\overline{\gamma})\nu(\boldsymbol{X} \bullet \boldsymbol{Y}) \qquad \text{(by 22)}, \\
&\ge -(1-\gamma)\nu(\boldsymbol{X} \bullet \boldsymbol{Y}).
\end{aligned}$$

Hence, for any sufficiently small positive σ, we have $\nu(\boldsymbol{X}(\sigma) \bullet \boldsymbol{Y}(\sigma)) - \nu(\boldsymbol{X} \bullet \boldsymbol{Y}) \ge -\sigma(1-\gamma)\nu(\boldsymbol{X} \bullet \boldsymbol{Y})$, *i.e.*, $\nu(\boldsymbol{X}(\sigma) \bullet \boldsymbol{Y}(\sigma)) \ge (1-(1-\gamma)\sigma)\,\theta$. ∎

Thus we construct an algorithm mapping $\mathcal{A}_{lg}$ from Ω into the collection of nonempty subsets of Ω such that

$$\mathcal{A}_{lg}(\boldsymbol{X}, \boldsymbol{Y}) = \{(\boldsymbol{X}(\sigma), \boldsymbol{Y}(\sigma)) : \sigma \in \Sigma\}$$

where $\Sigma \subset (0, 1]$ is a step length mapping. To complete an algorithm mapping $\mathcal{A}_{lg}$, we need to specify a step length rule to choose σ. Combining with the theorem above, we can use Armijo's rule ([31]) to determine the step length σ which meet our requirement 4 of Condition 5.2, since every data which construct our algorithm, such as ψ, $\mathcal{T}_k$, $(\boldsymbol{X}(\sigma), \boldsymbol{Y}(\sigma))$ are continuous w.r.t. $(\boldsymbol{X}, \boldsymbol{Y})$ and parameters.

Remark 5.6. In general, it is not easy to compute the next iterate $(\boldsymbol{X}(\sigma), \boldsymbol{Y}(\sigma))$ by equations (23), because it is not easy to compute the tangent space $\mathcal{L}_k$ and the curve $C(t)$ on $\mathcal{F}$ (especially, for the case that $\mathcal{F}$ is not affine). ∎

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Appendix – Proof of Lemma 5.3.

Suppose that Condition 5.2 holds. Then the sequence $\{(\boldsymbol{X}_k, \boldsymbol{Y}_k)\}$ satisfies one of the following conditions (see [25, Lemma 2.2]);

(A) $(\boldsymbol{X}_k, \boldsymbol{Y}_k) \in \Omega^*$ for some $k < \infty$.
(B) The sequence $\{(\boldsymbol{X}_k, \boldsymbol{Y}_k)\}$ is bounded and every limiting point of the sequence belongs to Ω^*.
(C) The sequence $\{(\boldsymbol{X}_k, \boldsymbol{Y}_k)\}$ is unbounded.

When (A) or (B) occurs, there is a finite number k for which (A)' holds. Therefore we assume (C) that the sequence $\{(\boldsymbol{X}_k, \boldsymbol{Y}_k)\}$ is unbounded.

(i) At each iteration, we have

$$\left.\begin{array}{l} (\boldsymbol{X}_k, \boldsymbol{Y}_k) \in \mathcal{S}_{++} \times \mathcal{S}_{++}, \\ (\bar{\boldsymbol{X}}_k, \bar{\boldsymbol{Y}}_k) = (\boldsymbol{X}_k, \boldsymbol{Y}_k) - \theta_k(\boldsymbol{B}_X, \boldsymbol{B}_Y) \in \mathcal{F}. \end{array}\right\}$$

By the monotonicity of $\mathcal{F}$, we then see that

$$\begin{array}{rcl} 0 & \leq & (\boldsymbol{X}_k - \theta_k \boldsymbol{B}_X - \bar{\boldsymbol{X}}_0) \bullet (\boldsymbol{Y}_k - \theta_k \boldsymbol{B}_Y - \bar{\boldsymbol{Y}}_0) \\ & = & (\boldsymbol{X}_k - \boldsymbol{X}_0 + (1-\theta_k)\boldsymbol{B}_X) \bullet (\boldsymbol{Y}_k - \boldsymbol{Y}_0 + (1-\theta_k)\boldsymbol{B}_Y) \\ & = & \boldsymbol{X}_k \bullet \boldsymbol{Y}_k - (\boldsymbol{X}_0 \bullet \boldsymbol{Y}_k + \boldsymbol{X}_k \bullet \boldsymbol{Y}_0) + \boldsymbol{X}_0 \bullet \boldsymbol{Y}_0 \\ & & +(1-\theta_k)\left((\boldsymbol{X}_k - \boldsymbol{X}_0) \bullet \boldsymbol{B}_Y + \boldsymbol{B}_X \bullet (\boldsymbol{Y}_k - \boldsymbol{Y}_0)\right) \\ & & +(1-\theta_k)^2 \boldsymbol{B}_X \bullet \boldsymbol{B}_Y; \end{array}$$

hence

$$\begin{array}{l} (1-\theta_k)(\boldsymbol{X}_k \bullet \boldsymbol{B}_Y + \boldsymbol{B}_X \bullet \boldsymbol{Y}_k) \\ \quad \geq \quad -\boldsymbol{X}_k \bullet \boldsymbol{Y}_k + (\boldsymbol{X}_0 \bullet \boldsymbol{Y}_k + \boldsymbol{X}_k \bullet \boldsymbol{Y}_0) - \boldsymbol{X}_0 \bullet \boldsymbol{Y}_0 \\ \qquad +(1-\theta_k)(\boldsymbol{X}_0 \bullet \boldsymbol{B}_Y + \boldsymbol{B}_X \bullet \boldsymbol{Y}_0) - (1-\theta_k)^2 \boldsymbol{B}_X \bullet \boldsymbol{B}_Y. \end{array}$$

When the sequence $\{(\boldsymbol{X}_k, \boldsymbol{Y}_k) \succ \boldsymbol{O}\}$ is unbounded, the term

$$(\boldsymbol{X}_0 \bullet \boldsymbol{Y}_k + \boldsymbol{X}_k \bullet \boldsymbol{Y}_0)$$

on the right hand side of the inequality diverges to $+\infty$. Since all other terms on the right hand side remains bounded by Condition 5.2, the right hand side diverges to $+\infty$; hence so does the left hand side. Thus we have shown that

$$(1-\theta_k)(\boldsymbol{X}_k \bullet \boldsymbol{B}_Y + \boldsymbol{B}_X \bullet \boldsymbol{Y}_k) \to \infty \quad \text{as } k \to \infty. \tag{26}$$

Hence we must have that $\boldsymbol{B} = (\boldsymbol{B}_X, \boldsymbol{B}_Y) \neq (\boldsymbol{O}, \boldsymbol{O})$. If $\theta_k \leq \epsilon$ for some $k < \infty$ then (B)' occurs. Otherwise, we have

$$\theta_k \geq \epsilon \quad \text{for every } k = 0, 1, \cdots .$$

This together with the relation (26) implies

$$\theta_k (\boldsymbol{X}_k \bullet \boldsymbol{B}_Y + \boldsymbol{B}_X \bullet \boldsymbol{Y}_k) \to \infty \quad \text{as } k \to \infty.$$

Therefore there is a k for which

$$-\boldsymbol{X}_k \bullet \boldsymbol{Y}_k + \theta_k(\boldsymbol{X}_k \bullet \boldsymbol{B}_Y + \boldsymbol{B}_X \bullet \boldsymbol{Y}_k) - (\theta_k)^2 \boldsymbol{B}_X \bullet \boldsymbol{B}_Y \geq \theta_k M$$

holds; hence (C)' occurs.

(ii) Suppose that $(\boldsymbol{X}, \boldsymbol{Y})$ is a solution of the SDCP (1). Then

$$\left.\begin{array}{l} (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F},\ (\boldsymbol{X}, \boldsymbol{Y}) \succeq \boldsymbol{O},\ \boldsymbol{X} \bullet \boldsymbol{Y} = 0 \\ (\bar{\boldsymbol{X}}_k, \bar{\boldsymbol{Y}}_k) = (\boldsymbol{X}_k, \boldsymbol{Y}_k) - \theta_k(\boldsymbol{B}_X, \boldsymbol{B}_Y) \in \mathcal{F}. \end{array}\right\}$$

By the monotonicity of $\mathcal{F}$, we have

$$\begin{array}{rcl} 0 & \leq & (\boldsymbol{X}_k - \theta_k \boldsymbol{B}_X - \boldsymbol{X}) \bullet (\boldsymbol{Y}_k - \theta_k \boldsymbol{B}_Y - \boldsymbol{Y}) \\ & = & \boldsymbol{X}_k \bullet \boldsymbol{Y}_k - \theta_k(\boldsymbol{X}_k \bullet \boldsymbol{B}_Y + \boldsymbol{B}_X \bullet \boldsymbol{Y}_k) + (\theta_k)^2 \boldsymbol{B}_X \bullet \boldsymbol{B}_Y \\ & & -(\boldsymbol{X}_k - \theta_k \boldsymbol{B}_X) \bullet \boldsymbol{Y} - \boldsymbol{X} \bullet (\boldsymbol{Y}_k - \theta_k \boldsymbol{B}_Y) + \boldsymbol{X} \bullet \boldsymbol{Y} \\ & \leq & \boldsymbol{X}_k \bullet \boldsymbol{Y}_k - \theta_k(\boldsymbol{X}_k \bullet \boldsymbol{B}_Y + \boldsymbol{B}_X \bullet \boldsymbol{Y}_k) + (\theta_k)^2 \boldsymbol{B}_X \bullet \boldsymbol{B}_Y \\ & & +\theta_k(\boldsymbol{B}_X \bullet \boldsymbol{Y} + \boldsymbol{X} \bullet \boldsymbol{B}_Y); \end{array}$$

hence

$$\theta_k(\boldsymbol{B}_X \bullet \boldsymbol{Y} + \boldsymbol{X} \bullet \boldsymbol{B}_Y) \geq -\boldsymbol{X}_k \bullet \boldsymbol{Y}_k + \theta_k(\boldsymbol{X}_k \bullet \boldsymbol{B}_Y + \boldsymbol{B}_X \bullet \boldsymbol{Y}_k) - (\theta_k)^2 \boldsymbol{B}_X \bullet \boldsymbol{B}_Y.$$

Therefore if (C)' occurs there is no solution of the SDCP (1) such that

$$(\boldsymbol{B}_X \bullet \boldsymbol{Y} + \boldsymbol{X} \bullet \boldsymbol{B}_Y) < M.$$

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