Existence of Search Directions in Interior-Point Algorithms for the SDP and the Monotone SDLCP

Masayuki Shida[†], Susumu Shindoh[‡], Masakazu Kojima[♯]

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Abstract. Various search directions used in interior-point-algorithms for the SDP (semidefinite program) and the monotone SDLCP (semidefinite linear complementarity problem) are characterized by the intersection of a maximal monotone affine subspace and a maximal and strictly antitone affine subspace. This observation provides a unified geometric view over the existence of those search directions.

Key words Interior-Point Algorithm, Semidefinite Program, Semidefinite Linear Complementarity Problem, Monotonicity

† Department of Mathematics, Kanagawa University,
Rokkakubashi 3-27-1, Kanagawa-ku, Yokohama 221, Japan.

‡ Department of Mathematics and Physics, The National Defense Academy,
Hashirimizu 1-10-20, Yokosuka, 239, Japan.

♯ Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, 2-12-1 Oh-Okayama, Meguro-ku, Tokyo 152, Japan.

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Tokyo Institute of Technology, Tokyo, Japan

1. Introduction.

The primal-dual interior-point method is known as one of the most efficient computational methods for solving large scale linear programs. There are numerous variations of the method, but most of them basically utilize a common search direction, which was introduced in the original primal-dual interior-point method [8, 18] and is often interpreted as the Newton direction towards a point on the central trajectory. By the primal-dual search direction, we mean this common search direction when we are concerned with interior-point methods for linear programs. The primal-dual search direction was extended naturally to the monotone linear complementarity problem. See [9, 10, etc.]. Several distinct search directions have been proposed, however, in recent primal-dual interior-point methods [1, 2, 6, 14, 16, 21, etc.] for the SDP (semidefinite program) and interior-point methods [12] for the monotone SDLCP (semidefinite linear complementarity problem). Many of them may be regarded as extensions of the primal-dual search direction used for linear programs, but they are essentially different.

The purpose of this short note is to give a characterization of those search directions proposed for the SDP and the monotone SDLCP in terms of the intersection of a maximal monotone affine subspace and a strictly and maximal antitone affine subspace. This characterization provides us not only a simple proof of their unique existence but also a unified geometric view over those search directions. We deal with five kinds of search directions; one search direction from the paper [16] for the SDP (semidefinite program), one from [14] for the SDP, one from [2, 6] for the SDP, one from [12] for the monotone SDLCP (semidefinite linear complementarity problem), and one from [2] for the SDP.

In Section 2, we state some basic definitions and general lemmas related to the concepts of monotonicity and antitonicity. In Sections 3, we present the main theorem (Theorem 3.1). Although we will restrict ourselves to the SDP and the monotone SDLCP, we can apply the lemmas and the characterization of search directions to other various problems (see remark (A) in Section 4).

2. Basic Lemmas.

Let $\mathcal{H}$ be a finite dimensional real vector space with the inner product $\boldsymbol{x} \cdot \boldsymbol{y}$ of $\boldsymbol{x} \in \mathcal{H}$ and $\boldsymbol{y} \in \mathcal{H}$. We call a subset $\mathcal{A}$ of $\mathcal{H} \times \mathcal{H}$ *monotone* (or *antitone*, resp.) if $(\boldsymbol{u} - \boldsymbol{u}') \cdot (\boldsymbol{v} - \boldsymbol{v}') \geq 0$ (or ≤ 0, resp.) for every pair $(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{A}$ and $(\boldsymbol{u}', \boldsymbol{v}') \in \mathcal{A}$. If the strict inequality holds for any distinct pair $(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{A}$ and $(\boldsymbol{u}', \boldsymbol{v}') \in \mathcal{A}$, we call $\mathcal{A}$ *strictly monotone* (or *strictly antitone*, resp.). We say that a monotone (or antitone, resp.) subset $\mathcal{A}$ is *maximal*, if there is no monotone (or antitone, resp.) subset of $\mathcal{H} \times \mathcal{H}$ which properly contains $\mathcal{A}$. See [3, 5, 13, etc.].

We state three lemmas. Lemma 2.1 and Corollary 2.2 give some characterizations of a maximal monotone (or maximal antitone) affine subspace of $\mathcal{H} \times \mathcal{H}$. Lemma 2.3 plays a key role in proving the existence and uniqueness of search directions in Section 3.

Lemma 2.1. *Let $\mathcal{A}$ be a monotone (or antitone, resp.) affine subspace of $\mathcal{H} \times \mathcal{H}$. Then $\mathcal{A}$ is maximal monotone (maximal antitone, resp.) if and only if* $\dim \mathcal{A} = \dim \mathcal{H}$.

Proof: Although the assertion of the lemma is well-known (see *e.g.* [5]), we include an elementary proof here. Since we know that an affine subspace $\mathcal{A}$ of $\mathcal{H} \times \mathcal{H}$ is (strictly

and/or maximal, resp.) monotone if and only if $\{(\boldsymbol{x}, -\boldsymbol{y}) \in \mathcal{H} \times \mathcal{H} \ : \ (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{A}\}$ is (strictly and/or maximal, resp.) antitone, we only consider the case where $\mathcal{A}$ is a monotone affine subspace of $\mathcal{H} \times \mathcal{H}$. Let $m = \dim \mathcal{H}$. Choose an $(\boldsymbol{x}^0, \boldsymbol{y}^0) \in \mathcal{A}$ and let $\mathcal{A}_0 = \{(\boldsymbol{x} - \boldsymbol{x}^0, \boldsymbol{y} - \boldsymbol{y}^0) : (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{A}\}$. Then $\mathcal{A}_0$ is a monotone linear subspace of $\mathcal{H} \times \mathcal{H}$. It suffices to prove that $\mathcal{A}_0$ is maximal monotone if and only if $\dim \mathcal{A}_0 = m$.

"if" part To prove the "if" part, let $\dim \mathcal{A}_0 = m$. Assume on the contrary that $\mathcal{A}_0$ is not maximal, *i.e.*, there is a nonzero vector $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}) \in (\mathcal{H} \times \mathcal{H}) \backslash \mathcal{A}_0$ such that $\mathcal{A}_0 + (\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}})$ is monotone subset in $\mathcal{H} \times \mathcal{H}$. Then we have for every $(\boldsymbol{u}^{A_0}, \boldsymbol{v}^{A_0}) \in \mathcal{A}_0$,

$$(\boldsymbol{u}^{A_0} - \bar{\boldsymbol{u}}) \cdot (\boldsymbol{v}^{A_0} - \bar{\boldsymbol{v}}) \geq 0. \tag{1}$$

Let $\bar{\mathcal{A}} := \mathcal{A}_0 + \mathcal{R}(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}) := \{(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}^{A_0}, \boldsymbol{v}^{A_0}) + r(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}) : (\boldsymbol{u}^{A_0}, \boldsymbol{v}^{A_0}) \in \mathcal{A}_0, r \in \mathcal{R}\}$. Then $\bar{\mathcal{A}}$ is an $(m+1)$-dimensional linear subspace of $\mathcal{H} \times \mathcal{H}$. We show the linear subspace $\bar{\mathcal{A}}$ is monotone. Let $(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}^{A_0}, \boldsymbol{v}^{A_0}) + r(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}})$ be an arbitrary element of $\bar{\mathcal{A}}$. If $r = 0$, we have $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^{A_0} \cdot \boldsymbol{v}^{A_0} \geq 0$ since $\mathcal{A}_0$ is a monotone subspace. If $r \neq 0$, we have

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{v} &= (\boldsymbol{u}^{A_0} + r\bar{\boldsymbol{u}}) \cdot (\boldsymbol{v}^{A_0} + r\bar{\boldsymbol{v}}) = r^2 \left(-\frac{\boldsymbol{u}^{A_0}}{r} - \bar{\boldsymbol{u}} \right) \cdot \left(-\frac{\boldsymbol{v}^{A_0}}{r} - \bar{\boldsymbol{v}} \right) \\ &\geq 0 \qquad \text{since } \mathcal{A}_0 \text{ is a linear subspace and inequality (1)} \end{aligned}$$

Therefore, the $(m+1)$-dimensional linear subspace $\bar{\mathcal{A}}$ is monotone. Let $\mathcal{B}_a = \{(\boldsymbol{w}, -\boldsymbol{w}) \in \mathcal{H} \times \mathcal{H} : \boldsymbol{w} \in \mathcal{H}\}$. It is easy to see that $\mathcal{B}_a$ is an m-dimensional strictly antitone linear subspace of $\mathcal{H} \times \mathcal{H}$. Since $\dim \bar{\mathcal{A}} + \dim \mathcal{B}_a = 2m + 1$, we have $\dim(\bar{\mathcal{A}} \cap \mathcal{B}_a) \geq 1$. This implies that there is a nonzero vector $(\boldsymbol{p}, -\boldsymbol{p}) \in \bar{\mathcal{A}} \cap \mathcal{B}_a$. This contradicts to the fact that $\bar{\mathcal{A}}$ is monotone.

"only if" part To prove the "only if" part, we show that there is an m-dimensional monotone linear subspace $\mathcal{D}$ of $\mathcal{H} \times \mathcal{H}$ containing $\mathcal{A}_0$ if $\dim \mathcal{A}_0 < m$. Let $\mathcal{B}_m = \{(\boldsymbol{w}, \boldsymbol{w}) : \boldsymbol{w} \in \mathcal{H}\}$ and $\mathcal{D} = \mathcal{A}_0 + \mathcal{B}_m \cap \mathcal{A}_0^{\perp}$, where $\mathcal{C}^{\perp}$ denotes the orthogonal complement of a linear subspace $\mathcal{C}$ of $\mathcal{H} \times \mathcal{H}$. Then $\mathcal{B}_m$ an m-dimensional strictly monotone linear subspace of $\mathcal{H} \times \mathcal{H}$ and $\mathcal{B}_m^{\perp} = \mathcal{B}_a$. Since $\mathcal{B}_a$ is strictly antitone and $\mathcal{A}_0$ is monotone, we have $\mathcal{B}_a \cap \mathcal{A}_0 = \{\boldsymbol{0}\}$. Hence

$$\begin{aligned} &\dim \left(\mathcal{B}_m \cap \mathcal{A}_0^{\perp} \right) = \dim(\mathcal{B}_a + \mathcal{A}_0)^{\perp} = 2m - (m + \dim \mathcal{A}_0) = m - \dim \mathcal{A}_0, \\ &\dim \mathcal{D} = \dim \left(\mathcal{A}_0 + \mathcal{B}_m \cap \mathcal{A}_0^{\perp} \right) = \dim \mathcal{A}_0 + \dim \left(\mathcal{B}_m \cap \mathcal{A}_0^{\perp} \right) = m. \end{aligned}$$

Thus we have shown that $\dim \mathcal{D} = m$. To show the monotonicity of $\mathcal{D}$, suppose that $(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{D}$. Then there exist $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{A}_0$ and $(\boldsymbol{w}, \boldsymbol{w}) \in \mathcal{B}_m \cap \mathcal{A}_0^{\perp}$ such that $(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{x} + \boldsymbol{w}, \boldsymbol{y} + \boldsymbol{w})$. It follows that

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{v} &= (\boldsymbol{x} + \boldsymbol{w}) \cdot (\boldsymbol{y} + \boldsymbol{w}) \\ &= \boldsymbol{x} \cdot \boldsymbol{y} + (\boldsymbol{x}, \boldsymbol{y}) \cdot (\boldsymbol{w}, \boldsymbol{w}) + \boldsymbol{w} \cdot \boldsymbol{w} \\ &= \boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{w} \cdot \boldsymbol{w} \ (\text{since } (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{A}_0 \text{ and } (\boldsymbol{w}, \boldsymbol{w}) \in \mathcal{A}_0^{\perp}) \\ &\geq 0 \ (\text{since } \mathcal{A}_0 \text{ is monotone}). \end{aligned}$$

∎

Corollary 2.2. *Let $\mathcal{H}$ be a finite dimensional real vector space. Suppose that $\mathcal{A}$ is an affine subspace of $\mathcal{H} \times \mathcal{H}$ given by $\mathcal{A} = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{H} \times \mathcal{H} \ : \ \boldsymbol{y} = \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{q}, \ \boldsymbol{x} \in \mathcal{H}\}$, where $\boldsymbol{Q}$ is a linear operator on $\mathcal{H}$ and $\boldsymbol{q} \in \mathcal{H}$. Then $\mathcal{A}$ is strictly and maximal monotone (or strictly and maximal antitone, resp.) if and only if $\boldsymbol{Q}$ is positive definite, i.e., $\boldsymbol{u} \cdot \boldsymbol{Q}\boldsymbol{u} > 0$ for every nonzero $\boldsymbol{u} \in \mathcal{H}$ (or negative definite, i.e., $\boldsymbol{u} \cdot \boldsymbol{Q}\boldsymbol{u} < 0$ for every nonzero $\boldsymbol{u} \in \mathcal{H}$, resp.).*

Proof: From the definition of $\mathcal{A}$, we know that $\dim \mathcal{A} = \dim \mathcal{H}$ (the fact does not depend whether the matrix $\boldsymbol{Q}$ is positive definite or not). By Lemma 2.1, $\mathcal{A}$ is maximal monotone (or maximal antitone, resp.) whenever it is monotone (or antitone, resp.). On the other hand, it is easily verified that $\mathcal{A}$ is strictly monotone (or strictly antitone, resp.) if and only if $\boldsymbol{Q}$ is positive definite (or negative definite, resp.). Thus, we conclude the assertion. ∎

Lemma 2.3. (i) *Assume that an affine subspace $\mathcal{B}$ of $\mathcal{H} \times \mathcal{H}$ is strictly and maximal antitone. Then, $\mathcal{A} \cap \mathcal{B}$ is a singleton for every maximal monotone affine subspace $\mathcal{A}$ of $\mathcal{H} \times \mathcal{H}$.*

(ii) *Assume that an affine subspace $\mathcal{B}$ of $\mathcal{H} \times \mathcal{H}$ does not contain a maximal and strictly antitone affine subspace. Then $\mathcal{A} \cap \mathcal{B} = \emptyset$ for some maximal monotone affine subspace $\mathcal{A}$ of $\mathcal{H} \times \mathcal{H}$.*

Proof: **Part (i).** Let $\mathcal{A}$ be a maximal monotone affine subspace of $\mathcal{H} \times \mathcal{H}$. If $\mathcal{A} \cap \mathcal{B}$ contains at least 2 points $(\boldsymbol{u}_1, \boldsymbol{v}_1), (\boldsymbol{u}_2, \boldsymbol{v}_2)$, we have

$$\begin{array}{rcll} 0 & \leq & (\boldsymbol{u}_1 - \boldsymbol{u}_2) \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_2) & \text{since } \mathcal{A} \text{ is monotone} \\ & < & 0 & \text{since } \mathcal{B} \text{ is strictly antitone,} \end{array}$$

which is a contradiction. We show that $\mathcal{A} \cap \mathcal{B}$ is nonempty. Let

$$\begin{array}{ll} \mathcal{A}_0 := \mathcal{A} - (\boldsymbol{u}^A, \boldsymbol{v}^A) & \text{for some fixed } (\boldsymbol{u}^A, \boldsymbol{v}^A) \in \mathcal{A}, \\ \mathcal{B}_0 := \mathcal{B} - (\boldsymbol{u}^B, \boldsymbol{v}^B) & \text{for some fixed } (\boldsymbol{u}^B, \boldsymbol{v}^B) \in \mathcal{B}. \end{array}$$

Then $\mathcal{A}_0$ is a maximal monotone linear subspace of $\mathcal{H} \times \mathcal{H}$, and $\mathcal{B}_0$ is a strictly and maximal antitone subspace of $\mathcal{H} \times \mathcal{H}$. By the argument in the proof of Lemma 2.1, we have $\mathcal{A}_0 \oplus \mathcal{B}_0 = \mathcal{H} \times \mathcal{H}$. Hence, there exist $(\boldsymbol{x}^{A_0}, \boldsymbol{y}^{A_0}) \in \mathcal{A}_0$ and $(\boldsymbol{x}^{B_0}, \boldsymbol{y}^{B_0}) \in \mathcal{B}_0$ such that $(\boldsymbol{u}^A, \boldsymbol{v}^A) - (\boldsymbol{u}^B, \boldsymbol{v}^B) = (\boldsymbol{x}^{A_0}, \boldsymbol{y}^{A_0}) + (\boldsymbol{x}^{B_0}, \boldsymbol{y}^{B_0})$. Therefore, we have

$$\mathcal{A} \ni -(\boldsymbol{x}^{A_0}, \boldsymbol{y}^{A_0}) + (\boldsymbol{u}^A, \boldsymbol{v}^A) = (\boldsymbol{x}^{B_0}, \boldsymbol{y}^{B_0}) + (\boldsymbol{u}^B, \boldsymbol{v}^B) \in \mathcal{B},$$

thus we have $\mathcal{A} \cap \mathcal{B} \neq \emptyset$.

Part (ii). Let $\mathcal{B}_0 := \mathcal{B} - (\boldsymbol{u}^B, \boldsymbol{v}^B)$ for some fixed $(\boldsymbol{u}^B, \boldsymbol{v}^B) \in \mathcal{B}$. Let $\mathcal{C}$ be a maximal linear subspace of $\mathcal{B}_0$ with strict antitone property, and $\mathcal{D}$ be its orthogonal complement in $\mathcal{B}_0$. By the assumption and the construction, $\mathcal{C}$ is a strictly antitone subspace of $\mathcal{H} \times \mathcal{H}$ with $0 \leq \dim \mathcal{C} < \dim \mathcal{H}$ and $\mathcal{D}$ is a monotone subspace of $\mathcal{H} \times \mathcal{H}$ with $0 \leq \dim \mathcal{D} \leq \dim \mathcal{H}$. Let $\mathcal{A}_0$ be a maximal monotone linear subspace which contains $\mathcal{D}$. Then $\dim(\mathcal{A}_0 + \mathcal{B}_0) = \dim(\mathcal{A}_0 \oplus \mathcal{C}) = \dim \mathcal{A}_0 + \dim \mathcal{C} = \dim \mathcal{H} + \dim \mathcal{C} < \dim(\mathcal{H} \times \mathcal{H})$. Hence we can take a point $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \in \mathcal{H} \times \mathcal{H}$ such that $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \notin (\mathcal{A}_0 + \mathcal{B}_0)$. Now let $\mathcal{A} = \{(\boldsymbol{x} + \tilde{\boldsymbol{x}} + \boldsymbol{u}^B, \boldsymbol{y} + \tilde{\boldsymbol{y}} + \boldsymbol{v}^B) \ : \ (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{A}_0\}$. By the construction, $\mathcal{A}$ is a maximal monotone affine subspace of $\mathcal{H} \times \mathcal{H}$. If $(\boldsymbol{x}', \boldsymbol{y}') \in \mathcal{A} \cap \mathcal{B}$, we would have that

$$\mathcal{B} \ \ni \ \ (\boldsymbol{x}', \boldsymbol{y}') = (\boldsymbol{x} + \tilde{\boldsymbol{x}} + \boldsymbol{u}^B, \boldsymbol{y} + \tilde{\boldsymbol{y}} + \boldsymbol{v}^B) \ \text{ for some } (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{A}_0;$$

hence

$$(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = -(\boldsymbol{x}, \boldsymbol{y}) + (\boldsymbol{x}' - \boldsymbol{u}^B, \boldsymbol{y}' - \boldsymbol{v}^B) \in \mathcal{A}_0 + \mathcal{B}_0.$$

This contradicts $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \notin (\mathcal{A}_0 + \mathcal{B}_0)$. Thus we have shown that $\mathcal{A} \cap \mathcal{B} = \emptyset$. ∎

3. Main Results.

We first introduce notation and symbols which we use throughout this section. Let $\mathcal{M}$ denote the n^2-dimensional linear space of $n \times n$ real matrices, $\mathcal{S}$ the $n(n+1)/2$-dimensional linear space of $n \times n$ real symmetric matrices, and $\widetilde{\mathcal{S}}$ the $n(n-1)/2$-dimensional linear space of $n \times n$ real skew-symmetric matrices. We use the notation Tr $\boldsymbol{A}$ for the trace of a matrix $\boldsymbol{A} \in \mathcal{M}$ and the notation $\boldsymbol{A} \bullet \boldsymbol{B} = \mathrm{Tr}\ \boldsymbol{A}^T\boldsymbol{B}$ for the inner product of $\boldsymbol{A} \in \mathcal{M}$ and $\boldsymbol{B} \in \mathcal{M}$. We also use the notation

$$\mathbf{vec}\ \boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_{\cdot 1} \\ \boldsymbol{A}_{\cdot 2} \\ \vdots \\ \boldsymbol{A}_{\cdot n} \end{pmatrix} \in R^{n^2},$$

where $\boldsymbol{A}_{\cdot j} \in R^n$ denotes jth column of an $n \times n$ matrix $\boldsymbol{A}$. Then we see that

$$\boldsymbol{A} \bullet \boldsymbol{B} = (\mathbf{vec}\ \boldsymbol{A}) \cdot (\mathbf{vec}\ \boldsymbol{B}) \ \text{ for every } \boldsymbol{A},\ \boldsymbol{B} \in \mathcal{M}.$$

Therefore we may identify $\mathcal{M}$ as the n^2-dimensional Euclidean space R^{n^2} with the standard inner product, $\mathcal{S}$ as an $n(n+1)/2$-dimensional linear subspace of R^{n^2}, and $\widetilde{\mathcal{S}}$ as an $n(n-1)/2$-dimensional linear subspace of R^{n^2}. We also see that $\widetilde{\mathcal{S}}$ is the orthogonal complement of $\mathcal{S} \subset \mathcal{M}$;

$$\mathcal{S} + \widetilde{\mathcal{S}} = \mathcal{M} \ \text{ and } \boldsymbol{A} \bullet \boldsymbol{B} = 0 \ \text{ for every } \boldsymbol{A} \in \mathcal{S} \ \text{ and } \boldsymbol{B} \in \widetilde{\mathcal{S}}. \tag{2}$$

Let $\mathcal{S}_+$ denote the set of positive semidefinite matrices in $\mathcal{S}$, and $\mathcal{S}_{++}$ the set of positive definite matrices in $\mathcal{S}$. Given an $n(n+1)/2$-dimensional affine subspace $\mathcal{F}$ of $\mathcal{S} \times \mathcal{S}$ satisfying the monotonicity

$$(\boldsymbol{X}' - \boldsymbol{X}) \bullet (\boldsymbol{Y}' - \boldsymbol{Y}) \geq 0 \ \text{ for every } (\boldsymbol{X}', \boldsymbol{Y}') \in \mathcal{F} \ \text{ and } (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}, \tag{3}$$

we consider the SDLCP (semidefinite linear complementarity problem):

$$\text{Find an } (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F} \text{ such that } \boldsymbol{X} \in \mathcal{S}_+,\ \boldsymbol{Y} \in \mathcal{S}_+ \ \text{ and } \boldsymbol{X} \bullet \boldsymbol{Y} = 0. \tag{4}$$

The monotone SDLCP was introduced by Kojima, Shindoh and Hara [12] as an extension of the monotone LCP (linear complementarity problem) and a mathematical model for the primal-dual pair of SDPs (semidefinite programs). They proposed a family of interior-point algorithms for solving the monotone SDLCP (4). Also many primal-dual interior-point algorithms [2, 14, 16, etc.] developed for the SDP can be easily extended to the monotone SDLCP (4). See [12] for a more detailed discussion of the relations between the primal-dual pair of SDPs and the monotone SDLCP.

We investigate five kinds of search directions used in interior-point algorithms for the SDP and the SDLCP. We describe them in terms of the monotone SDLCP (4), and characterize each of them as the intersection of a maximal monotone affine subspace of $\mathcal{S} \times \mathcal{S}$ and a strictly and maximal antitone affine subspace of $\mathcal{S} \times \mathcal{S}$.

Suppose that $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_{++} \times \mathcal{S}_{++}$ is the current iterate of an interior-point algorithm for the monotone SDLCP (4). Let $\mu \geq 0$. All the search directions share an affine subspace

$$\mathcal{G} = \{(d\boldsymbol{X}, d\boldsymbol{Y}) \in \mathcal{S} \times \mathcal{S} \ : \ (\boldsymbol{X} + d\boldsymbol{X}, \boldsymbol{Y} + d\boldsymbol{Y}) \in \mathcal{F}\}. \tag{5}$$

Let $\boldsymbol{S} = \boldsymbol{X}^{1/2}(\boldsymbol{X}^{1/2}\boldsymbol{Y}\boldsymbol{X}^{1/2})^{-1/2}\boldsymbol{X}^{1/2}$, and let $\widetilde{\mathcal{F}}$ be a maximal monotone linear subspace of $\widetilde{\mathcal{S}} \times \widetilde{\mathcal{S}}$. Define

$$\left.\begin{array}{rcll}
\mathcal{L}_1 &=& \left\{(d\boldsymbol{X}, d\boldsymbol{Y}) \in \mathcal{S} \times \mathcal{S} \ : \ \begin{array}{l} \boldsymbol{S}^{-1} d\boldsymbol{X} \boldsymbol{S}^{-1} + d\boldsymbol{Y} = \mu \boldsymbol{X}^{-1} - \boldsymbol{Y} \end{array} \right\}, \\
\mathcal{L}_2 &=& \left\{(d\boldsymbol{X}, d\boldsymbol{Y}) \in \mathcal{S} \times \mathcal{S} \ : \ \begin{array}{l} (\boldsymbol{X}^{1/2})^{-1}(\boldsymbol{X} d\boldsymbol{Y} + d\boldsymbol{X}\boldsymbol{Y})\boldsymbol{X}^{1/2} \\ +\boldsymbol{X}^{1/2}(d\boldsymbol{Y}\boldsymbol{X} + \boldsymbol{Y} d\boldsymbol{X})(\boldsymbol{X}^{1/2})^{-1} \\ = 2(\mu \boldsymbol{I} - \boldsymbol{X}^{1/2}\boldsymbol{Y}\boldsymbol{X}^{1/2}) \end{array} \right\}, \\
\mathcal{L}_3 &=& \left\{(d\boldsymbol{X}, d\boldsymbol{Y}) \in \mathcal{S} \times \mathcal{S} \ : \ \begin{array}{l} d\boldsymbol{Y} + \mu \boldsymbol{X}^{-1} d\boldsymbol{X} \boldsymbol{X}^{-1} = \mu \boldsymbol{X}^{-1} - \boldsymbol{Y} \end{array} \right\}, \\
\mathcal{L}_4 &=& \left\{(d\boldsymbol{X}, d\boldsymbol{Y}) \in \mathcal{S} \times \mathcal{S} \ : \ \begin{array}{l} \boldsymbol{X}(d\boldsymbol{Y} + \widetilde{d\boldsymbol{Y}}) + (d\boldsymbol{X} + \widetilde{d\boldsymbol{X}})\boldsymbol{Y} \\ = \mu \boldsymbol{I} - \boldsymbol{X}\boldsymbol{Y} \\ \text{for some } (\widetilde{d\boldsymbol{X}}, \widetilde{d\boldsymbol{Y}}) \in \widetilde{\mathcal{F}} \end{array} \right\}, \\
\mathcal{L}_5 &=& \left\{(d\boldsymbol{X}, d\boldsymbol{Y}) \in \mathcal{S} \times \mathcal{S} \ : \ \begin{array}{l} \boldsymbol{X} d\boldsymbol{Y} + d\boldsymbol{Y}\boldsymbol{X} + d\boldsymbol{X}\boldsymbol{Y} + \boldsymbol{Y} d\boldsymbol{X} \\ = 2\mu \boldsymbol{I} - (\boldsymbol{X}\boldsymbol{Y} + \boldsymbol{Y}\boldsymbol{X}) \end{array} \right\}.
\end{array}\right\} \tag{6}$$

Then a vector $(d\boldsymbol{X}, d\boldsymbol{Y})$ in the intersection of $\mathcal{G}$ with each $\mathcal{L}_i$ $(i = 1, 2, 3, 4, 5)$ corresponds to the search direction proposed and studied in the papers [16], [14], [2], [12], and [2], respectively.

The theorem below ensures the existence and the uniqueness of each of these search directions, *i.e.*, $\mathcal{G} \cap \mathcal{L}_i$ is a singleton $(i = 1, 2, 3, 4, 5)$. But we should mention that for $i = 1, 2, 3, 4$ the existence and the uniqueness of the search direction corresponding to $\mathcal{G} \cap \mathcal{L}_i$ is already known. See the papers [16], [14], [2, 6] and [12], respectively.

Recall that $\widetilde{\mathcal{F}}$ in the definition of $\mathcal{L}_4$ is an arbitrary maximal monotone linear subspace of $\widetilde{\mathcal{S}} \times \widetilde{\mathcal{S}}$. Hence $\mathcal{G} \cap \mathcal{L}_4$ defines a family of search directions [12]. It is known that the search direction given in the paper [14], which we have described as the unique element in $\mathcal{G} \cap \mathcal{L}_2$, belongs to this family. More precisely, $\mathcal{G} \cap \mathcal{L}_2 = \mathcal{G} \cap \mathcal{L}_4$ holds if we take $\widetilde{\mathcal{F}} = \{\boldsymbol{O}\} \times \widetilde{\mathcal{S}}$. In this case, we can characterize the search direction $(d\boldsymbol{X}, d\boldsymbol{Y}) \in \mathcal{G} \cap \mathcal{L}_4$ as the unique solution of

$$\boldsymbol{X}(d\boldsymbol{Y} + \widetilde{d\boldsymbol{Y}}) + d\boldsymbol{X}\boldsymbol{Y} = \mu \boldsymbol{I} - \boldsymbol{X}\boldsymbol{Y} \ \text{ and } (d\boldsymbol{X}, d\boldsymbol{Y}) \in \mathcal{G} \ \text{ for some } \ \widetilde{d\boldsymbol{Y}} \in \widetilde{\mathcal{S}}.$$

This search direction was proposed independently by two group of researchers, Helmberg-Rendl-Vanderbei-Wolkowicz [6] and Kojima-Shindoh-Hara [12].

The search direction which we have described as the unique element of $\mathcal{G} \cap \mathcal{L}_5$ was proposed by Alizadeh, Haeberly and Overton [2]. We call it the AHO search direction here. They presented an interior-point method using the AHO search direction for the SDP and reported through numerical results that their method worked very efficiently. However, neither the existence of the AHO search direction nor theoretical convergence of their interior-point method with the use of the AHO direction was discussed in the paper [2]. We will see in the theorem below that the AHO search direction uniquely exists whenever the current iterate $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_{++} \times \mathcal{S}_{++}$ satisfies $(\boldsymbol{X}\boldsymbol{Y} + \boldsymbol{Y}\boldsymbol{X}) \in \mathcal{S}_+$. The paper [11] studies the global and local convergence of a predictor-corrector interior-point method using the AHO search direction for the monotone SDLCP.

Theorem 3.1. *Let $\mathcal{F}$ be a maximal monotone affine subspace of $\mathcal{S}\times\mathcal{S}$, and $\mathcal{G}$ be the affine subspace defined in (5). Assume that $(\boldsymbol{X},\boldsymbol{Y})\in\mathcal{S}_{++}\times\mathcal{S}_{++}$ and $\mu\geq 0$. Define the affine subspaces $\mathcal{L}_i$ $(i=1,2,\ldots,5)$ as in (6).*

(i) *$\mathcal{G}$ is a maximal monotone affine subspace of $\mathcal{S}\times\mathcal{S}$.*

(ii) *For $i=1,2,3,4$, $\mathcal{L}_i$ is a strictly and maximal antitone affine subspace of $\mathcal{S}\times\mathcal{S}$, and $\mathcal{G}\cap\mathcal{L}_i$ is a singleton.*

(iii) *If $(\boldsymbol{XY}+\boldsymbol{YX})\in\mathcal{S}_+$, then $\mathcal{L}_5$ is a strictly and maximal antitone affine subspace of $\mathcal{S}\times\mathcal{S}$ and $\mathcal{G}\cap\mathcal{L}_5$ is a singleton.*

Remark 3.2. The requirement $(\boldsymbol{XY}+\boldsymbol{YX})\in\mathcal{S}_+$ in (iii) of Theorem 3.1 holds if $(\boldsymbol{X},\boldsymbol{Y})$ lies in some neighborhood of the central surface

$$\{(\boldsymbol{X},\boldsymbol{Y})\in\mathcal{S}_{++}\times\mathcal{S}_{++}:\boldsymbol{XY}=\mu\boldsymbol{I}\ \text{ for some }\mu>0\}$$

such as

$$\left\{(\boldsymbol{X},\boldsymbol{Y})\in\mathcal{S}_{++}\times\mathcal{S}_{++}:\left\|\frac{1}{2}(\boldsymbol{XY}+\boldsymbol{YX})-\mu\boldsymbol{I}\right\|_F\leq\gamma\mu\ \text{ for some }\mu>0\right\}\ \text{ or}$$
$$\{(\boldsymbol{X},\boldsymbol{Y})\in\mathcal{S}_{++}\times\mathcal{S}_{++}:\boldsymbol{XY}+\boldsymbol{YX}\succeq(1-\gamma)\mu\boldsymbol{I}\ \text{ for some }\mu>0\},$$

where $\gamma\in[0,1)$. See also the paper [11].

Let $\boldsymbol{A}\otimes\boldsymbol{B}$ denote the Kronecker product of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. In the proof of Theorem 3.1 below, we will utilize the following lemma repeatedly. We also use the relation $(\boldsymbol{A}\otimes\boldsymbol{B})\mathbf{vec}\ \boldsymbol{M}=\mathbf{vec}\ (\boldsymbol{BMA}^T)$. See *e.g.* [4] for its proof and some other basic properties on the Kronecker product of matrices.

Lemma 3.3. *Let λ_i $(i=1,2,\ldots,n)$ and μ_j $(j=1,2,\ldots,n)$ be the eigenvalues of $\boldsymbol{A}\in\mathcal{M}$ and $\boldsymbol{B}\in\mathcal{M}$, respectively. Then $\lambda_i\mu_j$ $(i=1,2,\ldots,n,\ j=1,2,\ldots,n)$ are the eigenvalues of $\boldsymbol{A}\otimes\boldsymbol{B}$. If in addition $\boldsymbol{A}\in\mathcal{S}_{++}$ and $\boldsymbol{B}\in\mathcal{S}_{++}$, then $\boldsymbol{A}\otimes\boldsymbol{B}$ is symmetric and positive definite.*

Proof of Theorem 3.1 : The assertion (i) follows directly from the maximal monotonicity of $\mathcal{F}$. To prove the assertions (ii) and (iii), we rewrite the affine subspaces $\mathcal{L}_i$ $(i=1,2,3,4,5)$ using the Kronecker product as follows:

$$\left.\begin{array}{rcl}\mathcal{L}_i &=& \{(d\boldsymbol{X},d\boldsymbol{Y})\in\mathcal{S}\times\mathcal{S}\ :\ \mathbf{vec}\ d\boldsymbol{Y}=\boldsymbol{Q}_i\mathbf{vec}\ d\boldsymbol{X}+\boldsymbol{q}_i\}\ (i=1,2,3,5),\\ \mathcal{L}_4 &=& \left\{(d\boldsymbol{X},d\boldsymbol{Y})\in\mathcal{S}\times\mathcal{S}\ :\ \begin{array}{l}\mathbf{vec}\ (d\boldsymbol{Y}+\widetilde{d\boldsymbol{Y}})=\boldsymbol{Q}_4\mathbf{vec}\ (d\boldsymbol{X}+\widetilde{d\boldsymbol{X}})+\boldsymbol{q}_4\\ \text{for some }(\widetilde{d\boldsymbol{X}},\widetilde{d\boldsymbol{Y}})\in\widetilde{\mathcal{F}}\end{array}\right\},\end{array}\right\}\quad(7)$$

where

$$\left.\begin{array}{rcl}\boldsymbol{Q}_1 &=& -\boldsymbol{S}^{-1}\otimes\boldsymbol{S}^{-1},\ \boldsymbol{q}_1=\mathbf{vec}\ (\mu\boldsymbol{X}^{-1}-\boldsymbol{Y}),\\ \boldsymbol{Q}_2 &=& -\dfrac{\boldsymbol{Y}\otimes\boldsymbol{X}^{-1}+\boldsymbol{X}^{-1}\otimes\boldsymbol{Y}}{2},\ \boldsymbol{q}_2=\mathbf{vec}\ (\mu\boldsymbol{X}^{-1}-\boldsymbol{Y}),\\ \boldsymbol{Q}_3 &=& -\mu\boldsymbol{X}^{-1}\otimes\boldsymbol{X}^{-1},\ \boldsymbol{q}_3=\mathbf{vec}\ (\mu\boldsymbol{X}^{-1}-\boldsymbol{Y}),\\ \boldsymbol{Q}_4 &=& -\boldsymbol{Y}\otimes\boldsymbol{X}^{-1},\ \boldsymbol{q}_4=\mathbf{vec}\ \left(\mu\boldsymbol{X}^{-1}-\boldsymbol{Y})\right),\\ \boldsymbol{Q}_5 &=& -(\boldsymbol{I}\otimes\boldsymbol{X}+\boldsymbol{X}\otimes\boldsymbol{I})^{-1}(\boldsymbol{I}\otimes\boldsymbol{Y}+\boldsymbol{Y}\otimes\boldsymbol{I}),\\ \boldsymbol{q}_5 &=& \mathbf{vec}\ (2\mu\boldsymbol{I}-(\boldsymbol{XY}+\boldsymbol{YX}))\,.\end{array}\right\}\quad(8)$$

(Note that the nonsingularity of the matrix $(\boldsymbol{I} \otimes \boldsymbol{X} + \boldsymbol{X} \otimes \boldsymbol{I})$ follows from $\boldsymbol{X} \in \mathcal{S}_{++}$ and Lemma 3.3). We know from the definition of $\mathcal{L}_i$ $(i = 1, 2, 3)$ in (6) that

$$d\boldsymbol{Y} \in \mathcal{S} \ \text{ if } \mathbf{vec}\ d\boldsymbol{Y} = \boldsymbol{Q}_i(\mathbf{vec}\ d\boldsymbol{X}) + \boldsymbol{q}_i \ \text{ and } d\boldsymbol{X} \in \mathcal{S}\ (i = 1, 2, 3);$$

hence we may regard $\boldsymbol{Q}_i$ as a linear operator on $\mathcal{S}$. We also see from $\boldsymbol{X} \in \mathcal{S}_{++}$, $\boldsymbol{Y} \in \mathcal{S}_{++}$, $\boldsymbol{S} \in \mathcal{S}_{++}$ and Lemma 3.3 that $\boldsymbol{Q}_i$ is negative definite on $\{\mathbf{vec}\ \boldsymbol{U} : \boldsymbol{U} \in \mathcal{S}\}$ $(i = 1, 2, 3)$. Thus we obtain by Lemma 2.2 that $\mathcal{L}_i$ $(i = 1, 2, 3)$ are strictly and maximal antitone subspaces of $\mathcal{S} \times \mathcal{S}$.

Now we consider the case of $\mathcal{L}_4$. Define

$$\widehat{\mathcal{L}}_4(\boldsymbol{q}) = \{(\widehat{d\boldsymbol{X}}, \widehat{d\boldsymbol{Y}}) \in \mathcal{M} \times \mathcal{M} \ : \ \mathbf{vec}\ \widehat{d\boldsymbol{Y}} = \boldsymbol{Q}_4 \mathbf{vec}\ \widehat{d\boldsymbol{X}} + \boldsymbol{q}\} \ \text{ for every } \boldsymbol{q} \in R^{n \times n}. \quad (9)$$

Since $\boldsymbol{Q}_4$ is negative definite by Lemma 3.3, we know by Lemma 2.2 that $\widehat{\mathcal{L}}_4(\boldsymbol{q})$ is a strictly and maximal antitone affine subspace of $\mathcal{M} \times \mathcal{M}$ for every $\boldsymbol{q} \in \boldsymbol{R}^{n \times n}$. Using this fact we will show that $\mathcal{L}_4$ is a strictly antitone affine subspace of $\mathcal{S} \times \mathcal{S}$. Suppose that $(d\boldsymbol{X}_1, d\boldsymbol{Y}_1) \in \mathcal{L}_4$, $(d\boldsymbol{X}_2, d\boldsymbol{Y}_2) \in \mathcal{L}_4$ and $(d\boldsymbol{X}_1, d\boldsymbol{Y}_1) \neq (d\boldsymbol{X}_2, d\boldsymbol{Y}_2)$. Then there exists an $(\widetilde{d\boldsymbol{X}}, \widetilde{d\boldsymbol{Y}}) \in \widetilde{\mathcal{F}}$ such that $(d\boldsymbol{X}_1 - d\boldsymbol{X}_2 + \widetilde{d\boldsymbol{X}}, d\boldsymbol{Y}_1 - d\boldsymbol{Y}_2 + \widetilde{d\boldsymbol{Y}}) \in \widehat{\mathcal{L}}_4(\boldsymbol{0})$. Since $(d\boldsymbol{X}_1 - d\boldsymbol{X}_2 + \widetilde{d\boldsymbol{X}}, d\boldsymbol{Y}_1 - d\boldsymbol{Y}_2 + \widetilde{d\boldsymbol{Y}}) \neq \boldsymbol{O}$ and $\widehat{\mathcal{L}}_4(\boldsymbol{0})$ is a strictly and maximal antitone linear subspace of $\mathcal{M} \times \mathcal{M}$, we have that

$$\begin{aligned}
0 \ &> \ (d\boldsymbol{X}_1 - d\boldsymbol{X}_2 + \widetilde{d\boldsymbol{X}}) \bullet (d\boldsymbol{Y}_1 - d\boldsymbol{Y}_2 + \widetilde{d\boldsymbol{Y}}) \\
&= \ (d\boldsymbol{X}_1 - d\boldsymbol{X}_2) \bullet (d\boldsymbol{Y}_1 - d\boldsymbol{Y}_2) + \widetilde{d\boldsymbol{X}} \bullet \widetilde{d\boldsymbol{Y}} \\
&\quad\ (\text{since } (d\boldsymbol{X}_1 - d\boldsymbol{X}_2),\ (d\boldsymbol{Y}_1 - d\boldsymbol{Y}_2) \in \mathcal{S} \text{ and } \widetilde{d\boldsymbol{X}},\ \widetilde{d\boldsymbol{Y}} \in \widetilde{\mathcal{S}} \text{ (see (2))} \\
&\geq \ (d\boldsymbol{X}_1 - d\boldsymbol{X}_2) \bullet (d\boldsymbol{Y}_1 - d\boldsymbol{Y}_2) \ \text{ (since } \widetilde{\mathcal{F}} \text{ is monotone).}
\end{aligned}$$

Thus we have shown that $\mathcal{L}_4$ is a strictly antitone affine subspace of $\mathcal{S} \times \mathcal{S}$. To derive its maximality in $\mathcal{S} \times \mathcal{S}$, we show that for every $\overline{d\boldsymbol{X}} \in \mathcal{S}$ there exists a unique $\overline{d\boldsymbol{Y}} \in \mathcal{S}$ for which $(\overline{d\boldsymbol{X}}, \overline{d\boldsymbol{Y}}) \in \mathcal{L}_4$; this implies $\dim \mathcal{L}_4 = \dim \mathcal{S}$, hence the maximality in $\mathcal{S} \times \mathcal{S}$ follows from Lemma 2.1. Let $\overline{d\boldsymbol{X}} \in \mathcal{S}$ be fixed. Consider

$$\mathcal{K} = \{(\overline{d\boldsymbol{X}} + \widetilde{d\boldsymbol{X}}, d\boldsymbol{Y} + \widetilde{d\boldsymbol{X}}) \in \mathcal{M} \times \mathcal{M} \ : \ d\boldsymbol{Y} \in \mathcal{S},\ (\widetilde{d\boldsymbol{X}}, \widetilde{d\boldsymbol{Y}}) \in \widetilde{\mathcal{F}}\}.$$

Then $\mathcal{K}$ forms an n^2-dimensional monotone affine subspace of $\mathcal{M} \times \mathcal{M}$. Applying Lemma 2.3, we obtain that there exists a unique $(\overline{d\boldsymbol{X}} + \widetilde{d\boldsymbol{X}}, \overline{d\boldsymbol{Y}} + \widetilde{d\boldsymbol{X}}) \in \mathcal{K} \cap \widehat{\mathcal{L}}_4(\boldsymbol{q}_4)$, which implies that $(\overline{d\boldsymbol{X}}, \overline{d\boldsymbol{Y}}) \in \mathcal{L}_4$.

We now deal with the case of $\mathcal{L}_5$. If a pair of $d\boldsymbol{X} \in \mathcal{S}$ and $d\boldsymbol{Y} \in \mathcal{M}$ satisfies

$$\boldsymbol{X}d\boldsymbol{Y} + d\boldsymbol{Y}\boldsymbol{X} + d\boldsymbol{X}\boldsymbol{Y} + \boldsymbol{Y}d\boldsymbol{X} = 2\mu\boldsymbol{I} - (\boldsymbol{X}\boldsymbol{Y} + \boldsymbol{Y}\boldsymbol{X}),$$

then the pair of $d\boldsymbol{X} \in \mathcal{S}$ and $d\boldsymbol{Y}^T \in \mathcal{M}$ does. But we can rewrite this identity as $\mathbf{vec}\ d\boldsymbol{Y} = \boldsymbol{Q}_5 \mathbf{vec}\ d\boldsymbol{X} + \boldsymbol{q}_5$. Hence $d\boldsymbol{Y} = d\boldsymbol{Y}^T \in \mathcal{M}$ if $\mathbf{vec}\ d\boldsymbol{Y} = \boldsymbol{Q}_5 \mathbf{vec}\ d\boldsymbol{X} + \boldsymbol{q}_5$ and $d\boldsymbol{X} \in \mathcal{S}$. This implies that $\mathbf{vec}\ d\boldsymbol{Y} = \boldsymbol{Q}_5 \mathbf{vec}\ d\boldsymbol{X} + \boldsymbol{q}_5 \in \{\mathbf{vec}\ \boldsymbol{U} : \boldsymbol{U} \in \mathcal{S}\}$ for every $d\boldsymbol{X} \in \mathcal{S}$, and that $\dim \mathcal{L}_5 = \dim \mathcal{S}$. We have left to show that $\boldsymbol{Q}_5$ is negative definite on $\{\mathbf{vec}\ \boldsymbol{U} : \boldsymbol{U} \in \mathcal{S}\}$. Let $\boldsymbol{O} \neq \boldsymbol{U} \in \mathcal{S}$ and $\boldsymbol{W}$ be the nonzero matrix in $\mathcal{S}$ uniquely determined by $\boldsymbol{X}\boldsymbol{W} + \boldsymbol{W}\boldsymbol{X} = \boldsymbol{U}$, *i.e.*, $\mathbf{vec}\ \boldsymbol{W} = (\boldsymbol{I} \otimes \boldsymbol{X} + \boldsymbol{X} \otimes \boldsymbol{I})^{-1} \mathbf{vec}\ \boldsymbol{U})$. Then

$$\mathbf{vec}\ \boldsymbol{U} \cdot (\boldsymbol{Q}_5 \mathbf{vec}\ \boldsymbol{U})$$

$$
\begin{aligned}
&= -\mathbf{vec}\, \boldsymbol{U} \cdot \left((\boldsymbol{I} \otimes \boldsymbol{X} + \boldsymbol{X} \otimes \boldsymbol{I})^{-1} (\boldsymbol{I} \otimes \boldsymbol{Y} + \boldsymbol{Y} \otimes \boldsymbol{I}) \mathbf{vec}\, \boldsymbol{U} \right) \\
&= -\mathbf{vec}\, \boldsymbol{W} \cdot ((\boldsymbol{I} \otimes \boldsymbol{Y} + \boldsymbol{Y} \otimes \boldsymbol{I})(\boldsymbol{I} \otimes \boldsymbol{X} + \boldsymbol{X} \otimes \boldsymbol{I}) \mathbf{vec}\, \boldsymbol{W}) \\
&= -\mathbf{vec}\, \boldsymbol{W} \cdot ((\boldsymbol{X} \otimes \boldsymbol{Y} + \boldsymbol{Y} \otimes \boldsymbol{X} + \boldsymbol{Y}\boldsymbol{X} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{Y}\boldsymbol{X}) \mathbf{vec}\, \boldsymbol{W}) \\
&< -\mathbf{vec}\, \boldsymbol{W} \cdot ((\boldsymbol{Y}\boldsymbol{X} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{Y}\boldsymbol{X}) \mathbf{vec}\, \boldsymbol{W}) \\
&\quad \text{(since } \boldsymbol{X} \otimes \boldsymbol{Y} \text{ and } \boldsymbol{Y} \otimes \boldsymbol{X} \text{ are positive definite by Lemma 3.3)} \\
&= -\boldsymbol{W} \bullet (\boldsymbol{W}\boldsymbol{X}\boldsymbol{Y} + \boldsymbol{Y}\boldsymbol{X}\boldsymbol{W}) \\
&= -\mathrm{Tr}\, \boldsymbol{W} (\boldsymbol{X}\boldsymbol{Y} + \boldsymbol{Y}\boldsymbol{X}) \boldsymbol{W} \\
&\leq 0.
\end{aligned}
$$

Here the lase inequality follows from the assumption that $\boldsymbol{X}\boldsymbol{Y} + \boldsymbol{Y}\boldsymbol{X} \in \mathcal{S}_+$.

Finally we obtain by Lemma 2.3 that $\mathcal{G} \cap \mathcal{L}_i$ is a singleton ($i = 1, 2, 3, 4, 5$). This completes the proof of Theorem 3.1. ∎

4. Concluding Remarks on Theorem 3.1.

(A) For each $\mu > 0$, define

$$\mathcal{C}(\mu) = \{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_+ \times \mathcal{S}_+ \ : \ \boldsymbol{X}\boldsymbol{Y} = \mu \boldsymbol{I}\}.$$

We can prove that $\mathcal{C}(\mu)$ is a strictly and maximal antitone smooth manifold in $\mathcal{S} \times \mathcal{S}$ for every μ. Each point of the central trajectory of the monotone SDLCP (4) is uniquely determined by the intersection of the maximal monotone affine subspace $\mathcal{F}$ and the strictly and maximal antitone affine subspace $\mathcal{C}(\mu)$. Our characterization of five types of search directions used in the SDP and the monotone SDLCP (4) is founded on this fact. The systems of linear equations involved in $\mathcal{L}_i$ ($i = 2, 3, 4, 5$) (see (6)) were derived as distinct (scaled) linear approximations of the system of equations $\boldsymbol{X}\boldsymbol{Y} = \mu \boldsymbol{I}$ at $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_{++} \times \mathcal{S}_{++}$ ([1, 12, 14]), while the system of linear equations involved in $\mathcal{L}_1$ (see (6)) was derived from the self-concordant barrier function ([15, 16]). When $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_{++} \times \mathcal{S}_{++}$ lies on $\mathcal{C}(\mu)$, all $\mathcal{L}_i$ ($i = 1, 2, 3, 4, 5$) coincide with the tangent space of $\mathcal{C}(\mu)$ at $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{C}(\mu)$,

$$
\begin{aligned}
&\{(d\boldsymbol{X}, d\boldsymbol{Y}) \in \mathcal{S} \times \mathcal{S} : \boldsymbol{X} d\boldsymbol{Y} + d\boldsymbol{X}\boldsymbol{Y} = \boldsymbol{O}\} \\
&\quad = \{(d\boldsymbol{X}, d\boldsymbol{Y}) \in \mathcal{S} \times \mathcal{S} : \mu \mathbf{vec}\, d\boldsymbol{Y} = -(\boldsymbol{Y} \otimes \boldsymbol{Y}) \mathbf{vec}\, d\boldsymbol{X}\},
\end{aligned}
$$

which inherits the strict and maximal antitonicity from $\mathcal{C}(\mu)$. It is shown in the recent paper [17] that each point on the central trajectory of the monotone generalized complementarity problem [7] is characterized as the intersection of a maximal monotone subset associated with the feasible region and a strictly and a maximal monotone subset, as in the case of the SDLCP. Applying the basic lemmas given in Section 2, we can derive the similar characterization on some other search directions used in interior-point methods ([8, 9, 16, 20, etc.]) for various problems covered by the monotone generalized complementarity problem.

(B) The strict and maximal antitonicity of the affine subspace $\mathcal{L}_i$ of $\mathcal{S} \times \mathcal{S}$ is necessary to ensure $\mathcal{G} \cap \mathcal{L}_i \neq \emptyset$ for any maximal monotone affine subspace $\mathcal{F} \subset \mathcal{S} \times \mathcal{S}$ ($i = 1, 2, 3, 4, 5$). In other words, if the affine subspace $\mathcal{L}_i$ does not contain a strictly and maximal antitone affine subspace, then by (ii) of Lemma 2.3, there would be a maximal monotone affine subspace $\mathcal{F}$ and an $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_{++} \times \mathcal{S}_{++}$ such that $\mathcal{G} \cap \mathcal{L}_i = \emptyset$, where $\mathcal{G} = \{(d\boldsymbol{X}, d\boldsymbol{Y}) \in \mathcal{S} \times \mathcal{S} :$

$(\boldsymbol{X}+d\boldsymbol{X}, \boldsymbol{Y}+d\boldsymbol{Y}) \in \mathcal{F}\}$.

(C) The assumption $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_{++} \times \mathcal{S}_{++}$ does not imply $(\boldsymbol{XY}+\boldsymbol{XY}) \in \mathcal{S}_{++}$ in (iii) of Theorem 3.1. To see this, consider the pair of matrices

$$\boldsymbol{X} = \begin{pmatrix} 49 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{S}_{++} \quad \text{and } \boldsymbol{Y} = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{S}_{++}.$$

In this example, we have that

$$\boldsymbol{XY}+\boldsymbol{YX} = \begin{pmatrix} 392 & 50 \\ 50 & 2 \end{pmatrix} \notin \mathcal{S}_{+}.$$

Furthermore the matrix

$$\boldsymbol{Q}_5 = -(\boldsymbol{I}\otimes\boldsymbol{X}+\boldsymbol{X}\otimes\boldsymbol{I})^{-1}(\boldsymbol{I}\otimes\boldsymbol{Y}+\boldsymbol{Y}\otimes\boldsymbol{I}) = -\begin{pmatrix} 4/49 & 1/98 & 1/98 & 0 \\ 1/50 & 1/10 & 0 & 1/50 \\ 1/50 & 0 & 1/10 & 1/50 \\ 0 & 1/2 & 1/2 & 1 \end{pmatrix}$$

is not negative definite on $\{\mathbf{vec}\ \boldsymbol{U} : \boldsymbol{U} \in \mathcal{S}\}$ since we have

$$(\mathbf{vec}\ \boldsymbol{W})^T \boldsymbol{Q}_5 \mathbf{vec}\ \boldsymbol{W} = \frac{7}{5} > 0 \quad \text{for } \boldsymbol{W} = \begin{pmatrix} 0 & -5 \\ -5 & 2 \end{pmatrix} \in \mathcal{S}.$$

This implies that $\mathcal{L}_5$ is not antitone, then $\mathcal{L}_5$ does not contain maximal antitone affine subspace. In view of Remark (B) above, we can take a maximal monotone affine subspace $\mathcal{F} \in \mathcal{S}\times\mathcal{S}$ and an $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_{++} \times \mathcal{S}_{++}$ such that $\mathcal{L}_5 \cap \mathcal{G} = \emptyset$. See also the paper [19].

(D) The proof of the assertion that $\mathcal{L}_4$ is a maximal antitone affine subspace of $\mathcal{S}\times\mathcal{S}$ is a little bit complicated. If we are only concerned with the assertion that $\mathcal{G}\cap\mathcal{L}_4$ is a singleton, we can give a simple proof. Define $\widehat{\mathcal{L}}_4(\boldsymbol{q})$ by (9) and $\widehat{\mathcal{G}}$ by

$$\widehat{\mathcal{G}} = \{(d\boldsymbol{X}+\widetilde{d\boldsymbol{X}}, d\boldsymbol{Y}+\widetilde{d\boldsymbol{Y}}) \ :\ (d\boldsymbol{X}, d\boldsymbol{Y}) \in \mathcal{G},\ (\widetilde{d\boldsymbol{X}}, \widetilde{d\boldsymbol{Y}}) \in \tilde{\mathcal{F}}\},$$

then $\widehat{\mathcal{L}}_4(\boldsymbol{q}_4)$ is a strictly and maximal antitone affine subspace of $\mathcal{M}\times\mathcal{M}$, and $\widehat{\mathcal{G}}$ a maximal monotone affine subspace of $\mathcal{M}\times\mathcal{M}$. By Lemma 2.3, there exists a unique $(\widehat{d\boldsymbol{X}}^*, \widehat{d\boldsymbol{Y}}^*) \in \widehat{\mathcal{G}}\cap\widehat{\mathcal{L}}_4(\boldsymbol{q}_4)$. On the other hand, we see by construction that

$(d\boldsymbol{X}, d\boldsymbol{Y}) \in \mathcal{G}\cap\mathcal{L}_4$
if and only if $(d\boldsymbol{X}+\widetilde{d\boldsymbol{X}}, d\boldsymbol{Y}+\widetilde{d\boldsymbol{Y}}) \in \widehat{\mathcal{L}}_4(\boldsymbol{q}_4)$ for some $(\widetilde{d\boldsymbol{X}}, \widetilde{d\boldsymbol{Y}}) \in \tilde{\mathcal{F}}\times\tilde{\mathcal{F}}$.

Therefore $\mathcal{G}\cap\mathcal{L}_4$ is a singleton.

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