Centers of Monotone Generalized Complementarity Problems

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Abstract. Let C be a full dimensional, closed, pointed and convex cone in a finite dimensional real vector space $\mathcal{E}$ with an inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ of $\boldsymbol{x},\ \boldsymbol{y} \in \mathcal{E}$, and $\mathcal{M}$ a maximal monotone subset of $\mathcal{E} \times \mathcal{E}$. This paper studies the existence and continuity of centers of the monotone generalized complementarity problem associated with C and $\mathcal{M}$: Find $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{M} \cap (C \times C^*)$ such that $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$. Here $C^* = \{\boldsymbol{y} \in \mathcal{E} : \langle \boldsymbol{x}, \boldsymbol{y} \rangle \geq 0 \ \text{ for all } \boldsymbol{x} \in C\}$ denotes the dual cone of C. The main result of the paper unifies and extends some results established for monotone complementarity problems in Euclidean space and monotone semidefinite linear complementarity problems in symmetric matrices.

Key words Central Trajectory, Path of Centers, Complementarity Problem, Interior Point Algorithm, Linear Program

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1 Introduction.

The central trajectory or the path of centers is known to be an important concept which plays a substantial role in the development of interior-point methods for various problems such as linear programming problems (LPs) [11, 16, etc.], monotone linear and nonlinear complementarity problems (LCPs and NLCPs) [4, 9, 10, etc.], semidefinite programming problems (SDPs) [1, 19, etc.] and monotone semidefinite linear complementarity problems (SDLCPs) [12]. A more fundamental concept, the method of centers was introduced by Huard [7] for convex constrained sets and by Sonnevend [18] for polytopes, in connection with optimization problems. It was extended by Megiddo [15] to the central trajectory of an LP and to the central trajectory of a monotone LCP. In his paper [15], Megiddo showed that the central trajectory runs through the interior of the primal-dual feasible region of the LP (or the interior of the feasible region of the monotone LCP) and converges to a pair of primal-dual optimal solutions of the LP (or a solution of the LCP, respectively). Numerous interior-point algorithms are based on the idea of "numerically tracing the central trajectory;" on the theoretical side, many papers have studied the existence and continuity of the central trajectory for various problems ([4, 10, 12, 13, 15, etc.]). **The aim of this paper is a unification and a further extension of some of these theoretical studies on the central trajectory for the monotone generalized complementarity problem (GCP).**

Let $\mathcal{E}$ be a finite dimensional real vector space with an inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ of $\boldsymbol{x},\ \boldsymbol{y} \in \mathcal{E}$. A subset $\mathcal{M}$ of $\mathcal{E} \times \mathcal{E}$ is called *monotone* if $\langle \boldsymbol{x}' - \boldsymbol{x}, \boldsymbol{y}' - \boldsymbol{y} \rangle \geq 0$ for every $(\boldsymbol{x}', \boldsymbol{y}')$, $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{M}$. A monotone subset $\mathcal{M}$ of $\mathcal{E} \times \mathcal{E}$ is called *maximal* if there is no monotone subset of $\mathcal{E} \times \mathcal{E}$ which properly contains $\mathcal{M}$. Throughout the paper we assume:

- C is a full dimensional, closed, pointed and convex cone in $\mathcal{E}$; and
- $\mathcal{M}$ is a maximal monotone subset of $\mathcal{E} \times \mathcal{E}$.

Let $C^* = \{\boldsymbol{y} \in \mathcal{E} : \langle \boldsymbol{x}, \boldsymbol{y} \rangle \geq 0 \ \text{ for all } \boldsymbol{x} \in C\}$ denote the dual cone of C. Then C^* is also a full dimensional, closed, pointed and convex cone in $\mathcal{E}$, and $(C^*)^* = C$. This paper deals with the monotone generalized complementarity problem in $\mathcal{E}$ ([8, etc.]),

$$\textbf{GCP}: \quad \text{Find} \quad (\boldsymbol{x}, \boldsymbol{y}) \in C \times C^* \ \text{ such that } (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{M} \ \text{ and } \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0. \tag{1}$$

This problem is also called a monotone complementarity problem associated with a convex cone C.

Our succeeding discussion relies much on *a θ-logarithmically homogeneous self-concordant barrier for a cone*, which was originally introduced by Nesterov and Nemirovskii [17]. Let $\mathcal{R}$ denote the set of real numbers.

Definition 1.1. (Definitions 2.1.1, 2.3.2 and 2.4.1 of [17]): Let $\theta > 0$ and $F : \mathcal{E} \to \mathcal{R} \cup \{\infty\}$ be a proper convex function. F is said to be *self-concordant* on Int C if F is a C^3-function whose effective domain $\mathbf{dom} F = \{\boldsymbol{x} \in \mathcal{E} : F(\boldsymbol{x}) < \infty\}$ is Int C, the interior of C, and $|D^3 F(\boldsymbol{x})[\boldsymbol{h}, \boldsymbol{h}, \boldsymbol{h}]| \leq 2(D^2 F(\boldsymbol{x})[\boldsymbol{h}, \boldsymbol{h}])^{\frac{3}{2}}$ for all $\boldsymbol{x} \in$ Int C and all $\boldsymbol{h} \in \mathcal{E}$, where $D^k F(\boldsymbol{x})[\boldsymbol{h}_1, \cdots, \boldsymbol{h}_k]$ denotes the value of the k-th differential of F taken at $\boldsymbol{x}$ along the collection of directions $\boldsymbol{h}_1, \cdots, \boldsymbol{h}_k \in \mathcal{E}$. F is called a *θ-logarithmically homogeneous self-concordant barrier* for C if

F is self-concordant on Int C,
$F(\boldsymbol{x}^p) \to \infty$ for each sequence $\{\boldsymbol{x}^p \in$ Int $C\}$ that converges to the boundary of C,
$F(t\boldsymbol{x}) = F(\boldsymbol{x}) - \theta \log t$ for each $t > 0$ and $\boldsymbol{x} \in$ Int C.

The *modified Legendre transformation* $F^+ : \mathcal{E} \to \mathcal{R} \cup \{\infty\}$ of a θ-logarithmically homogeneous self-concordant barrier F for C is defined by $F^+(\boldsymbol{y}) := \sup\{-\langle \boldsymbol{x}, \boldsymbol{y}\rangle - F(\boldsymbol{x}) : \boldsymbol{x} \in \mathcal{E}\}$. Note that the effective domain of F^+ is Int C^*, the interior of the dual cone C^* of C. ∎

In general, we know that:

- Given an arbitrary full dimensional, closed, pointed and convex cone C, there is a θ-logarithmically homogeneous self-concordant barrier F for C for some $\theta \geq 1$ ([17, Theorem 2.5.1 and Remark 2.5.1]). See [17, Chapters 2 and 5] and [5, Sections 4 and 6] for the construction of a θ-logarithmically homogeneous self-concordant barrier for C.
- If F is a θ-logarithmically homogeneous self-concordant barrier for C for some $\theta \geq 1$ then the modified Legendre transformation F^+ is a θ-logarithmically homogeneous self-concordant barrier for the dual cone C^* of C ([17, Theorem 2.4.1]).
- If F is a logarithmically homogeneous self-concordant barrier for the cone C, then F is strictly convex on Int C ([17, Proposition 2.3.5]).

Consider the problem;

$$\left.\begin{array}{ll} \text{minimize} & f_\mu(\boldsymbol{x}, \boldsymbol{y}) := \langle \boldsymbol{x}, \boldsymbol{y}\rangle + \mu(F(\boldsymbol{x}) + F^+(\boldsymbol{y})) \\ \text{subject to} & (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{M}. \end{array}\right\} \tag{2}$$

Let

$$H(\mu) = \begin{cases} \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{E} \times \mathcal{E} : -\boldsymbol{y} = \mu F'(\boldsymbol{x})\} & \text{for } \mu > 0 \\ \{(\boldsymbol{x}, \boldsymbol{y}) \in C \times C^* : \langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0\} & \text{for } \mu = 0, \end{cases} \tag{3}$$

where $F'(\boldsymbol{x})$ denotes the unique element of $\mathcal{E}$ such that $\langle \boldsymbol{h}, F'(\boldsymbol{x})\rangle = DF(\boldsymbol{x})[\boldsymbol{h}]$ for $\boldsymbol{x}, \boldsymbol{h} \in \mathcal{E}$. Now we state our main result.

Theorem 1.2. *Assume that*

$$\mathcal{M} \cap (\mathit{Int}\ C \times \mathcal{E}) \neq \emptyset \ \mathit{and}\ \mathcal{M} \cap (C \times \mathit{Int}\ C^*) \neq \emptyset. \tag{4}$$

Then:

(i) *For every $\mu \geq 0$, the set $\mathcal{M} \cap H(\mu)$ is the set of global minimizers of (2), and is nonempty.*

(ii) *The set $\mathcal{M} \cap H(\mu)$ is upper semi-continuous as a point-to-set map from $\mathcal{R}_+ \to \mathcal{E} \times \mathcal{E}$, where $\mathcal{R}_+$ denotes the set of nonnegative real numbers. Especially, for every $\mu > 0$, the set $\mathcal{M} \cap H(\mu)$ consists of a single point $(\boldsymbol{x}(\mu), \boldsymbol{y}(\mu)) \in \mathit{Int}\ C \times \mathit{Int}\ C^*$, and hence is continuous w.r.t. μ.*

(iii) *The solution set of the GCP (1) is nonempty, compact and convex.*

For every $\mu > 0$, we call $(\boldsymbol{x}(\mu), \boldsymbol{y}(\mu))$ a center, and $\{(\boldsymbol{x}(\mu), \boldsymbol{y}(\mu)) : \mu > 0\}$ the central trajectory. Theorem 1.2 generalizes some results on the existence and continuity of certain trajectories in the monotone LCP [15], the monotone NLCP [4, 13] and the monotone SDLCP [12]. Assertion (ii) implies that the central trajectory is a continuous path in the interior of the feasible region and that as $\mu \searrow 0$ every accumulation point lies in the solution set of the GCP (1).

It should be noted that the assumption (4) seems slightly weaker than the assumption

$$\mathcal{M} \cap (\text{Int } C \times \text{Int } C^*) \neq \emptyset, \tag{4}'$$

which was imposed in most of the literature on the central trajectory ([9, 12, 13, 15, etc.]). But assertion (ii) implies (4)'. Hence, we know as a corollary of the theorem that (4)' and (4) are equivalent. See also [4].

In assertion (iii), the nonemptiness and compactness of the solution set of the GCP (1), which is equal to the set $\mathcal{M} \cap H(0)$, is easily derived by assertion (i) and the proof of (ii). See also [8]. For the interesting relation between the feasibility and the solvability for the GCP (1), see for example [3, Theorems 3.1, 7.1 and Example 5.1] and [14].

Our proof of Theorem 1.2 given in the next section is substantially based on the proofs of Theorems 2.1 and 3.1 of the paper [4] by Güler. If we define a point-to-set mapping $T : \mathcal{E} \to 2^{\mathcal{E}}$ by $T(\boldsymbol{x}) \equiv \{\boldsymbol{y} \in \mathcal{E} : (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{M}\}$ for every $\boldsymbol{x} \in \mathcal{E}$, then T is a maximal monotone operator on $\mathcal{E}$ and we can rewrite the GCP (1) as:

$$\text{Find } (\boldsymbol{x}, \boldsymbol{y}) \in C \times C^* \text{ such that } \boldsymbol{y} \in T(\boldsymbol{x}) \text{ and } \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0. \qquad (1)'$$

Thus we can describe the monotone GCP in terms of a maximal monotone operator T on $\mathcal{E}$ as well as a maximal monotone subset $\mathcal{M}$ of $\mathcal{E} \times \mathcal{E}$. The GCP of the form (1)' might be more convenient to utilize Güler's proofs of Theorems 2.1 and 3.1 of [4] effectively because he dealt with the monotone NLCP of the form:

$$\text{Find } (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{R}^n_+ \times \mathcal{R}^n_+ \text{ such that } \boldsymbol{y} \in T(\boldsymbol{x}) \text{ and } \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0,$$

where T is a maximal monotone operator on $\mathcal{R}^n$ and $\mathcal{R}^n_+$ the nonnegative orthant of $\mathcal{R}^n$. However, the monotone operator is often not explicitly given in practice, *e.g.*, the primal-dual pair of an LP and the Karush-Kuhn-Tucker condition for a convex program. Therefore, we translate Güler's proof in terms of a maximal monotone set $\mathcal{M}$ of $\mathcal{E} \times \mathcal{E}$ and adapt it to the GCP (1).

2 Proof of Theorem 1.2

In this section, we shall prove the main result (Theorem 1.2) of this paper.

The following lemmas are keys to the existence of the center.

Lemma 2.1.

(1). [2, Theorem 6.7.13]: *If $\mathcal{M}_1$ and $\mathcal{M}_2$ are maximal monotone and $\mathbf{0} \in$ Int $(P_x(\mathcal{M}_1) - P_x(\mathcal{M}_2))$ then the operator sum $\mathcal{M}_1 \oplus_{op} \mathcal{M}_2 := \{(\boldsymbol{x}, \boldsymbol{y}_1 + \boldsymbol{y}_2) : (\boldsymbol{x}, \boldsymbol{y}_1) \in \mathcal{M}_1, (\boldsymbol{x}, \boldsymbol{y}_2) \in \mathcal{M}_2\}$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ is maximal monotone, where $P_x(\mathcal{M})$ is the projection of $\mathcal{M}$ to the $\boldsymbol{x}$-space, i.e., $P_x(\mathcal{M}) \equiv \{\boldsymbol{x} \in \mathcal{E} : (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{M}$ for some $\boldsymbol{y}\}$.*

(2). [2, Theorem 6.7.10]: *Let $\mathcal{A}$ be a maximal monotone set of $\mathcal{E} \times \mathcal{E}$ and K a subset of $\mathcal{E}$. Assume that, for every $\boldsymbol{y} \in K$, there exist $\boldsymbol{x}_y \in \mathcal{E}$ and $c \in \mathcal{R}$ satisfying $\langle \boldsymbol{x}' - \boldsymbol{x}_y, \boldsymbol{y}' - \boldsymbol{y} \rangle \geq c$ for every $(\boldsymbol{x}', \boldsymbol{y}') \in \mathcal{A}$. Then* Int $co(K) \subseteq$ Int $P_y(\mathcal{A})$. *Here $co(K)$ denotes a convex hull of K and $P_y(\mathcal{A})$ is the projection of $\mathcal{A}$ to the $\boldsymbol{y}$-space.* ∎

The following lemma shows a symmetric property of $H(\mu)$ w.r.t. $\boldsymbol{x}$ and $\boldsymbol{y}$.

Lemma 2.2. *For any $\mu > 0$, $H(\mu) = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{E} \times \mathcal{E} : -\boldsymbol{x} = \mu(F^+)'(\boldsymbol{y})\} \subset$ Int $C \times$ Int C^*.*

Proof: Let $(\boldsymbol{x}, \boldsymbol{y}) \in H(\mu)$, *i.e.*, $-\boldsymbol{y} = \mu F'(\boldsymbol{x})$. Then we have $(\boldsymbol{x}, \boldsymbol{y}) \in$ Int $C \times$ Int C^* and $\mu(F^+)'(\boldsymbol{y}) = \mu(F^+)'(-\mu F'(\boldsymbol{x})) = (\mu \cdot \frac{1}{\mu})(F^+)'(-F'(\boldsymbol{x})) = (F^+)'(-F'(\boldsymbol{x})) = -\boldsymbol{x}$ by [17, Proposition 2.3.4 and Theorem 2.4.1]. Similarly, we can prove the converse. ∎

For any $\mu > 0$, the derivative of the objective function $f_\mu(\boldsymbol{x}, \boldsymbol{y})$ of (2) is equal to $\mathbf{0}$ if and only if $(\boldsymbol{x}, \boldsymbol{y}) \in H(\mu)$ by Lemma 2.2. Noting that F and F^+ are convex, if the set $\mathcal{M} \cap H(\mu)$ is nonempty, we see that the set of global minimizers of (2) is equal to $\mathcal{M} \cap H(\mu)$. We can easily show that the optimal value is equal to $-\theta\mu \log \mu$ by [17, Proposition 2.3.4 and Theorem 2.4.1]. Therefore, the set $\{(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{M} : f_\mu(\boldsymbol{x}, \boldsymbol{y}) \leq -\theta\mu \log \mu + g(\mu), \langle \boldsymbol{x}, \boldsymbol{y} \rangle = \theta\mu\}$ forms a neighborhood of the central trajectory, where g is a function such that $g(\mu) : \mathcal{R}_{++} \to \mathcal{R}_{++}$ and $g(\mu) \searrow 0$ as $\mu \searrow 0$.

Lemma 2.3. *Let $\bar{H}(\mu) = \{(\boldsymbol{x}, \boldsymbol{y}) : (\boldsymbol{x}, -\boldsymbol{y}) \in H(\mu)\}$. For any $\mu \geq 0$, the set $\bar{H}(\mu)$ is maximal monotone. If in addition $\mu > 0$, $\bar{H}(\mu)$ is strictly monotone, i.e., we have $\langle \boldsymbol{x} - \boldsymbol{x}', \boldsymbol{y} - \boldsymbol{y}' \rangle > 0$ for any distinct elements $(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}', \boldsymbol{y}')$ in $\bar{H}(\mu)$.*

Proof: Since the set $\bar{H}(\mu)$ is the subdifferential of the proper closed convex function μF (or the indicator function $\delta(\cdot|C)$ for $\mu = 0$), then $\bar{H}(\mu)$ is maximal monotone [2, Proposition 6.7.6]. Strict monotonicity of $\bar{H}(\mu)$ for every $\mu > 0$ follows from the strict convexity of the logarithmically homogeneous self-concordant barrier F for C. ∎

Now, we are in a position to prove Theorem 1.2.

The Proof of Theorem 1.2.

(i) By the assumption, there exist $(\boldsymbol{x}_0, \boldsymbol{y}_0) \in \mathcal{M} \cap (\text{Int } C \times \mathcal{E})$ and $(\bar{\boldsymbol{x}}_0, \bar{\boldsymbol{y}}_0) \in \mathcal{M} \cap (C \times \text{Int } C^*)$. By Lemma 2.2, we only have to show the nonemptiness of $\mathcal{M} \cap H(\mu)$. For this purpose, we shall show that $\mathbf{0} \in P_y(\mathcal{M} \oplus_{op} \bar{H}(\mu))$ since $\mathcal{M} \cap H(\mu) \neq \emptyset \Leftrightarrow (\boldsymbol{x}, \mathbf{0}) \in \mathcal{M} \oplus_{op} \bar{H}(\mu)$ for some $\boldsymbol{x}$. Define $\widehat{\mathcal{M}} = \mathcal{M} \oplus_{op} \bar{H}(0)$. Then we have that $\widehat{\mathcal{M}}$ is maximal monotone by Lemmas 2.1 (1) and 2.3 and the fact that $\mathbf{0} \in \{\boldsymbol{x}_0\} - \text{Int } C \subset \text{Int } (P_x(\mathcal{M}) - P_x(\bar{H}(\mu)))$. It is easy to see that if $\mu \geq 0$, then $\bar{H}(\mu) = \bar{H}(0) \oplus_{op} \bar{H}(\mu)$ and $\mathcal{M} \oplus_{op} \bar{H}(\mu) = \mathcal{M} \oplus_{op} \bar{H}(0) \oplus_{op} \bar{H}(\mu) = \widehat{\mathcal{M}} \oplus_{op} \bar{H}(\mu)$. Using Lemma 2.1 (1) again, $\mathcal{M} \oplus_{op} \bar{H}(\mu) = \widehat{\mathcal{M}} \oplus_{op} \bar{H}(\mu)$ is maximal monotone. Hence, it suffices to show that $\mathbf{0} \in P_y\left(\widehat{\mathcal{M}} \oplus_{op} \bar{H}(\mu)\right)$ for every $\mu \geq 0$.

Now suppose that the following condition holds:

$$\left\{ \begin{array}{c} \text{for all } \boldsymbol{y} \in P_y(\widehat{\mathcal{M}}) + P_y(\bar{H}(\mu)), \text{ there exist } \boldsymbol{x}_y \in \mathcal{E} \text{ and } c \in \mathcal{R}, \text{ satisfying} \\ \langle \boldsymbol{x}' - \boldsymbol{x}_y, \boldsymbol{y}' - \boldsymbol{y} \rangle \geq c \text{ for every } (\boldsymbol{x}', \boldsymbol{y}') \in \widehat{\mathcal{M}} \oplus_{op} \bar{H}(\mu). \end{array} \right. \tag{5}$$

Applying Lemma 2.1 (2) with $K = P_y(\widehat{\mathcal{M}}) + P_y(\bar{H}(\mu))$ and $\mathcal{A} = \widehat{\mathcal{M}} \oplus_{op} \bar{H}(\mu)$, we have $\text{Int } co\left(P_y(\widehat{\mathcal{M}}) + P_y(\bar{H}(\mu))\right) \subseteq \text{Int } P_y\left(\widehat{\mathcal{M}} \oplus_{op} \bar{H}(\mu)\right)$. Hence, we can conclude the inclusion

$$P_y\left(\widehat{\mathcal{M}} \oplus_{op} \bar{H}(\mu)\right) \supseteq \text{Int } \left(P_y(\widehat{\mathcal{M}}) + P_y(\bar{H}(\mu))\right) \supseteq \{\bar{\boldsymbol{y}}_0\} - \text{Int } C^* \ni \mathbf{0}.$$

It remains to show (5). Suppose $\boldsymbol{y} = \boldsymbol{y}_1 + \boldsymbol{y}_2 \in P_y(\widehat{\mathcal{M}}) + P_y(\bar{H}(\mu))$, with $\boldsymbol{y}_1 \in P_y(\widehat{\mathcal{M}})$ and $\boldsymbol{y}_2 \in P_y(\bar{H}(\mu))$. Put $\boldsymbol{x}_y$ such that $(\boldsymbol{x}_y, \boldsymbol{y}_1) \in \widehat{\mathcal{M}}$. Let $(\boldsymbol{x}', \boldsymbol{y}') = (\boldsymbol{x}', \boldsymbol{y}'_1 + \boldsymbol{y}'_2)$ be an arbitrary element in $\widehat{\mathcal{M}} \oplus_{op} \bar{H}(\mu)$ such that $(\boldsymbol{x}', \boldsymbol{y}'_1) \in \widehat{\mathcal{M}}$ and $(\boldsymbol{x}', \boldsymbol{y}'_2) \in \bar{H}(\mu)$. Then we have

$$\langle \boldsymbol{x}' - \boldsymbol{x}_y, \boldsymbol{y}'_1 - \boldsymbol{y}_1 \rangle \geq 0, \tag{6}$$

since $(\boldsymbol{x}', \boldsymbol{y}'_1)$, $(\boldsymbol{x}_y, \boldsymbol{y}_1)$ are elements of the monotone set $\widehat{\mathcal{M}}$. Since $(\boldsymbol{x}', \boldsymbol{y}'_2) \in \bar{H}(\mu)$, we have $\langle \boldsymbol{x}', \boldsymbol{y}'_2 \rangle = -\mu\theta$ (for $\mu > 0$, see [17, (2.3.13)]). Thus, we also have

$$\begin{array}{rcl} \langle \boldsymbol{x}' - \boldsymbol{x}_y, \boldsymbol{y}'_2 - \boldsymbol{y}_2 \rangle & = & \langle \boldsymbol{x}_y, \boldsymbol{y}_2 \rangle - \langle \boldsymbol{x}_y, \boldsymbol{y}'_2 \rangle - \langle \boldsymbol{x}', \boldsymbol{y}_2 \rangle + \langle \boldsymbol{x}', \boldsymbol{y}'_2 \rangle \\ & \geq & \langle \boldsymbol{x}_y, \boldsymbol{y}_2 \rangle - \mu\theta \end{array} \tag{7}$$

By (6) and (7), we see that the condition (5) is satisfied.

(ii) For every $\mu > 0$, the nonempty subset $\mathcal{M} \cap H(\mu)$ lies in Int $C \times$Int C^* by Lemma 2.2 and assertion (i). Uniqueness of $\mathcal{M} \cap H(\mu)$ for every $\mu > 0$ is easily derived by the monotonicity of $\mathcal{M}$ and the strict monotonicity of $\bar{H}(\mu)$ (see Lemma 2.3). Let $(\boldsymbol{x}_0, \boldsymbol{y}_0) \in \mathcal{M} \cap (\text{Int } C \times \text{Int } C^*)$ be fixed. We regard $\mathcal{M} \cap H(\cdot)$ as a point-to-set map from $\mathcal{R}_+ \to \mathcal{E} \times \mathcal{E}$. If the map is closed and uniformly compact, then the map is upper semi-continuous ([6, Theorem 3]). Therefore, we shall prove that the map $\mathcal{M} \cap H(\cdot)$ is closed and uniformly compact. For this purpose, it suffices to show that, for any compact subset K of $\mathcal{R}_+$, the set $\bigcup_{\mu \in K} (\mathcal{M} \cap H(\mu))$ is compact.

We first show the boundedness of $\bigcup_{\mu \in K} (\mathcal{M} \cap H(\mu))$. Set $\mu^* = \max\{\mu : \mu \in K\}$ and $d = \langle \boldsymbol{x}_0, \boldsymbol{y}_0 \rangle$. Suppose that $(\boldsymbol{x}, \boldsymbol{y}) \in \bigcup_{\mu \in K} (\mathcal{M} \cap H(\mu))$. Since $\mathcal{M}$ is monotone and $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \mu\theta$ ([17, Proposition 2.3.4]), we have $\langle \boldsymbol{x}, \boldsymbol{y}_0 \rangle + \langle \boldsymbol{x}_0, \boldsymbol{y} \rangle \leq \langle \boldsymbol{x}_0, \boldsymbol{y}_0 \rangle + \langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq d + \mu^*\theta$. Since $\boldsymbol{x}_0 \in \text{Int } C$ and $\boldsymbol{y}_0 \in \text{Int } C^*$, it is easy to see that the set $\{(\boldsymbol{x}, \boldsymbol{y}) \in C \times C^* : \langle \boldsymbol{x}, \boldsymbol{y}_0 \rangle + \langle \boldsymbol{x}_0, \boldsymbol{y} \rangle \leq d + \mu^*\theta\}$ is bounded. Therefore, we conclude the boundedness of the set $\bigcup_{\mu \in K} (\mathcal{M} \cap H(\mu))$.

Now we show the closedness of the set $\bigcup_{\mu \in K} (\mathcal{M} \cap H(\mu))$. Suppose that $\{(\boldsymbol{x}^q, \boldsymbol{y}^q)\}$ is a sequence of points in $\bigcup_{\mu \in K} (\mathcal{M} \cap H(\mu))$ converging to a point $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ where $(\boldsymbol{x}^q, \boldsymbol{y}^q) \in \mathcal{M} \cap H(\mu^q)$ for some $\mu^q \in K$. We may assume without loss of generality that $\{\mu^q\} \subset K$ converges to some $\bar{\mu} \in K$. It follows from $(\boldsymbol{x}^q, \boldsymbol{y}^q) \in \mathcal{M} \cap H(\mu^q)$ and $\{\mu^q\} \subset K \subset \mathcal{R}_+$ for every q that $(\boldsymbol{x}^q, \boldsymbol{y}^q) \in \mathcal{M} \cap (C \times C^*)$ for every $\boldsymbol{q}$. Since $\mathcal{M}$ is a maximal monotone subset of $\mathcal{E} \times \mathcal{E}$, it must be closed. Hence, $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \mathcal{M} \cap (C \times C^*)$. By the continuity of the inner product, we have $\mu^q = \dfrac{1}{\theta}\langle \boldsymbol{x}^q, \boldsymbol{y}^q \rangle \to \dfrac{1}{\theta}\langle \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}} \rangle = \bar{\mu} \in K \quad \text{as} \quad q \to \infty$. If $\bar{\mu} = 0$, then $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in H(0)$ because $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in C \times C^*$ and $\langle \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}} \rangle = 0$; hence $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \mathcal{M} \cap H(0) \subset \bigcup_{\mu \in K} (\mathcal{M} \cap H(\mu))$. Supposing $\bar{\mu} > 0$, we will show that $\bar{\boldsymbol{x}} \in \text{Int } C$. In this case, we may assume that $\mu^q \geq \epsilon > 0$ for every q for some positive constant $\epsilon > 0$, *i.e.*, the sequence $\{\mu^q\}$ is bounded and bounded away from zero. Then $\boldsymbol{y}^q = -\mu^q F'(\boldsymbol{x}^q)$ for every q. We assume that $\bar{\boldsymbol{x}}$ lies on the boundary of C. Then the sequence $\{F(\boldsymbol{x}^q)\}$ diverges to ∞ as $q \to \infty$ by the definition of the logarithmically homogeneous self-concordant barrier F. Note that $\{\mu^q\}$ is bounded and bounded away from zero, $\{\boldsymbol{x}^q\}$ converges to $\bar{\boldsymbol{x}}$ and F is a strictly convex function on Int C. Therefore, the sequence $\{\|F'(\boldsymbol{x}^q)\|\}$ (hence the sequence $\{\|\boldsymbol{y}^q\|\}$) diverges to ∞ as $q \to \infty$. This contradicts the boundedness of $\bigcup_{\mu \in K} (\mathcal{M} \cap H(\mu))$. Therefore, $\bar{\boldsymbol{x}} \in \text{Int } C$. By the continuity of $F'(\cdot)$ on Int C, we have $\bar{\boldsymbol{y}} = \lim_{q \to \infty} \boldsymbol{y}^q = \lim_{q \to \infty} -\mu^q F'(\boldsymbol{x}^q) = -\bar{\mu} F'(\bar{\boldsymbol{x}}) \in \text{Int } C^*$, which implies that $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in H(\bar{\mu})$. Hence, $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \mathcal{M} \cap H(\bar{\mu}) \subset \bigcup_{\mu \in K} (\mathcal{M} \cap H(\mu))$. Therefore, we have shown that $\bigcup_{\mu \in K} (\mathcal{M} \cap H(\mu))$ is closed.

(iii) Note that the set $\mathcal{M} \cap H(0)$ is the solution set of the GCP (1). By assertion (i) and the proof of (ii), it suffices to show the convexity of the solution set of the GCP (1). Letting $(\boldsymbol{x}_1, \boldsymbol{y}_1)$, $(\boldsymbol{x}_2, \boldsymbol{y}_2)$ be solutions of the GCP (1) and $0 \leq \lambda \leq 1$, we show that $(\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2, \lambda \boldsymbol{y}_1 + (1-\lambda)\boldsymbol{y}_2)$ is a solution of the the GCP (1). Since $(\boldsymbol{x}_1, \boldsymbol{y}_1)$ and $(\boldsymbol{x}_2, \boldsymbol{y}_2)$ both lie in the convex set $C \times C^*$, we first observe that

$$(\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2, \lambda \boldsymbol{y}_1 + (1-\lambda)\boldsymbol{y}_2) \in C \times C^*. \tag{8}$$

It follows that

$$\begin{array}{rcll} 0 & \leq & \langle \lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2, \lambda \boldsymbol{y}_1 + (1-\lambda)\boldsymbol{y}_2 \rangle & \\ & = & -\lambda(1-\lambda)\langle \boldsymbol{x}_2 - \boldsymbol{x}_1, \boldsymbol{y}_2 - \boldsymbol{y}_1 \rangle & (\text{since } \langle \boldsymbol{x}_1, \boldsymbol{y}_1 \rangle = \langle \boldsymbol{x}_2, \boldsymbol{y}_2 \rangle = 0) \\ & \leq & 0 & (\text{by the monotonicity of } \mathcal{M}). \end{array}$$

Hence we have shown the complementarity;

$$\langle \lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2, \lambda \boldsymbol{y}_1 + (1-\lambda)\boldsymbol{y}_2 \rangle = 0. \tag{9}$$

To show

$$(\boldsymbol{x}, \boldsymbol{y}) = (\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2, \lambda \boldsymbol{y}_1 + (1-\lambda)\boldsymbol{y}_2) \in \mathcal{M}, \tag{10}$$

we apply the following characterization of the maximal monotone set $\mathcal{M}$;

$\langle \boldsymbol{x} - \boldsymbol{u}, \boldsymbol{y} - \boldsymbol{v} \rangle \geq 0$ for all $(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{M} \Leftrightarrow (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{M}$, ([2, Proposition 6.7.2]).

Let $(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{M}$. By using the monotonicity of $\mathcal{M}$, we see that

$$\begin{array}{rcll} & & \langle (\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) - \boldsymbol{u}, (\lambda \boldsymbol{y}_1 + (1-\lambda)\boldsymbol{y}_2) - \boldsymbol{v} \rangle & \\ & = & -\langle \lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2, \boldsymbol{v} \rangle - \langle \boldsymbol{u}, \lambda \boldsymbol{y}_1 + (1-\lambda)\boldsymbol{y}_2 \rangle + \langle \boldsymbol{u}, \boldsymbol{v} \rangle & (\text{by (9)}) \\ & = & \lambda \langle \boldsymbol{x}_1 - \boldsymbol{u}, \boldsymbol{y}_1 - \boldsymbol{v} \rangle + (1-\lambda)\langle \boldsymbol{x}_2 - \boldsymbol{u}, \boldsymbol{y}_2 - \boldsymbol{v} \rangle & (\text{since } \langle \boldsymbol{x}_1, \boldsymbol{y}_1 \rangle = \langle \boldsymbol{x}_2, \boldsymbol{y}_2 \rangle = 0) \\ & \geq & 0 & (\text{monotonicity of } \mathcal{M}). \end{array}$$

Thus we have shown (10). By (8), (9) and (10), we conclude that $(\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2, \lambda \boldsymbol{y}_1 + (1-\lambda)\boldsymbol{y}_2)$ is a solution of the the GCP (1). ∎

3 Concluding Remarks.

The papers [4, 10, etc.] dealt with a "generalized" central trajectory for the monotone CP; the trajectory runs through infeasible but positive region and every accumulation point of the trajectory as $\mu \searrow 0$ lies in the solution set of the CP.

We can extend those results to the monotone GCP (1). Consider the family $\mathrm{GCP}_{\boldsymbol{r}}$ of monotone generalized complementarity problems:

$$\mathbf{GCP}_{\boldsymbol{r}}: \quad \text{Find an} \quad (\boldsymbol{x}, \boldsymbol{y}) \in C \times C^* \quad \text{such that} \quad (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{M}(\boldsymbol{r}), \qquad (1)''$$

where $\mathcal{M}(\boldsymbol{r}) := \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{E} \times \mathcal{E} : (\boldsymbol{x}, \boldsymbol{y}) - \boldsymbol{r} \in \mathcal{M}\}$ for every $\boldsymbol{r} \in \mathcal{E} \times \mathcal{E}$. Let

$$\mathcal{B}_{++} \quad := \quad \{\boldsymbol{r} \in \mathcal{E} \times \mathcal{E} : \mathcal{M}(\boldsymbol{r}) \cap (\mathrm{Int}\ C \times \mathrm{Int}\ C^*) \neq \emptyset\}.$$

Then the set $\mathcal{B}_{++}$ turns out to be an open and convex subset of $\mathcal{E} \times \mathcal{E}$. For every $\mu > 0$ and every $\boldsymbol{r} \in \mathcal{B}_{++}$, consider the problem:

$$\left.\begin{array}{ll} \text{minimize} & \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \mu(F(\boldsymbol{x}) + F^+(\boldsymbol{y})) \\ \text{subject to} & (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{M}(\boldsymbol{r}). \end{array}\right\} \qquad (2)''$$

Then the existence and uniqueness of the global minimizer $(\boldsymbol{x}(\mu, \boldsymbol{r}), \boldsymbol{y}(\mu, \boldsymbol{r}))(= \mathcal{M}(\boldsymbol{r}) \cap H(\mu))$ of the problem for each $(\mu, \boldsymbol{r}) \in \mathcal{R}_{++} \times \mathcal{B}_{++}$ follows from Theorem 1.2 and the fact that $\mathcal{M}(\boldsymbol{r})$ is maximal monotone, where $\mathcal{R}_{++}$ denotes the set of positive real numbers. Let $\boldsymbol{r}(\mu)$ be a continuous map from $\mathcal{R}_+$ into $\mathcal{E} \times \mathcal{E}$ such that $\boldsymbol{r}(0) = (\boldsymbol{0}, \boldsymbol{0})$. If $\boldsymbol{r}(\mu) \in \mathcal{B}_{++}$ for every $\mu \geq 0$, then $\{(\boldsymbol{x}(\mu, \boldsymbol{r}(\mu)), \boldsymbol{y}(\mu, \boldsymbol{r}(\mu))) : \mu > 0\}$ forms a continuous path whose accumulation point as $\mu \searrow 0$ lies in the solution set of the original GCP (1). We can prove this by using similar arguments as in [4, the proof of Theorem 4.3 (c)].

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