# Enclosing Ellipsoids and Elliptic Cylinders of Semialgebraic Sets and Their Application to Error Bounds in Polynomial Optimization

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Dedicated to Professor Yoshitsugu Yamamoto on the occasion of his 60th birthday

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# Outline

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1. Problem and Formulation
2. Applications to the Sensor Network Localization Problem with Exact Distance
3. Concluding Remarks

$\boldsymbol{M} \in \mathbb{S}^n_+$ ($n \times n$ positive semidefinite matrices, shape). Define

$$\varphi(\boldsymbol{x},\boldsymbol{c}) \equiv (\boldsymbol{x}-\boldsymbol{c})^T \boldsymbol{M} (\boldsymbol{x}-\boldsymbol{c}), \forall \boldsymbol{x} \in \mathbb{R}^n, \forall \boldsymbol{c} \in \mathbb{R}^n \text{ (center)},$$

Ellipsoidal set $E(\boldsymbol{c},\gamma) \equiv \{\boldsymbol{x} \in \mathbb{R}^n : \varphi(\boldsymbol{x},\boldsymbol{c}) \leq \gamma\}$, $\forall \gamma > 0$ (size).

> Problem: Given a semialgebraic set $F$, find a minimum (in $\gamma$) ellipsoidal set $E(\boldsymbol{c},\gamma)$ which encloses $F$.

Application to error bounds in Polynomial Optimization Problem

> POP : minimize $f_0(\boldsymbol{x})$ subject to $f_k(\boldsymbol{x}) \geq 0$ $(k = 1,2,\ldots,p)$.
> Here $f_k : \mathbb{R}^n \rightarrow \mathbb{R}$ : a polynomial $(k = 0,1,\ldots,p)$.

$\hat{\boldsymbol{x}}$ : an approx. opt. solution; $f_k(\hat{\boldsymbol{x}}) \geq 0$ $(k = 1,2,\ldots,p)$. Let

$$F = \{\boldsymbol{x} \in \mathbb{R}^n : f_k(\boldsymbol{x}) \geq 0, \ (k = 1,2,\ldots,p), \ f_0(\boldsymbol{x}) \leq f_0(\hat{\boldsymbol{x}})\}.$$

$F \subset E(\boldsymbol{c},\gamma) \Longrightarrow E(\boldsymbol{c},\gamma)$ contains $\hat{\boldsymbol{x}}$, all opt. sol. of POP.

> $\boldsymbol{M} = \boldsymbol{I} \Rightarrow \|\boldsymbol{x}-\boldsymbol{c}\|^2 \leq \gamma$ for $\forall$ opt. sol. $\boldsymbol{x}$
>
> $\boldsymbol{M} = \text{diag}(1,0,\ldots,0) \Rightarrow |x_1 - c_1|^2 \leq \gamma$ for $\forall$ opt. sol. $\boldsymbol{x}$

- Implemented in the Matlab software SparsePOP (Waki et.al) for solving POPs by the sparse SDP relaxation.

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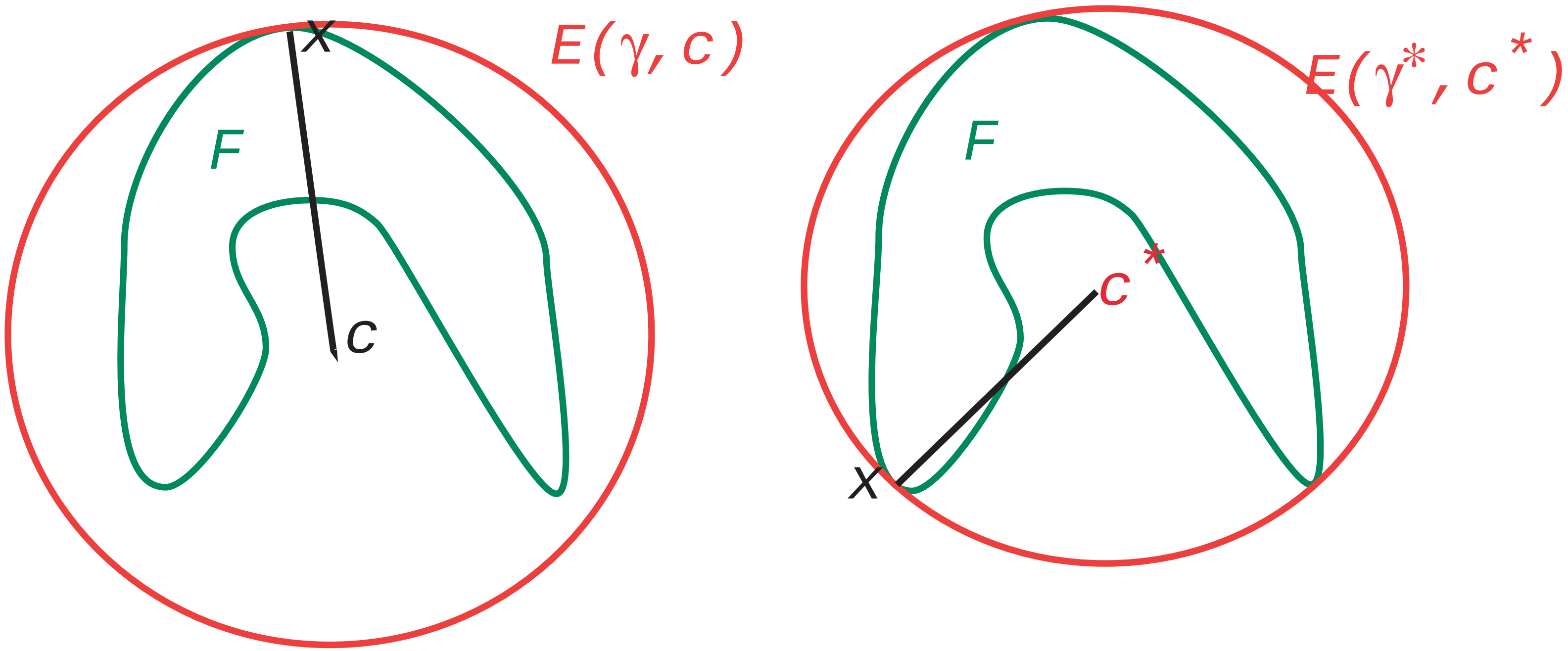
Ellipsoidal set $E(\boldsymbol{c}, \gamma) \equiv \{\boldsymbol{x} \in \mathbb{R}^n : \varphi(\boldsymbol{x}, \boldsymbol{c}) \leq \gamma\}, \ \forall \gamma > 0$ (size).

$F$ : a semialgebraic subset of $\mathbb{R}^n$.

min-max formulation

$$\gamma^* = \min_{\boldsymbol{c} \in \mathbb{R}^n} \max_{\boldsymbol{x} \in F} \varphi(\boldsymbol{x}, \boldsymbol{c}) = \max_{\boldsymbol{x} \in F} \varphi(\boldsymbol{x}, \boldsymbol{c}^*).$$

Suppose that $\boldsymbol{M} =$ the $2 \times 2$ identity matrix



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Lifting $\Rightarrow$

$\widehat{C} \equiv$ the convex hull of $\{(\boldsymbol{x}, \boldsymbol{W}) = (\boldsymbol{x}, \boldsymbol{x}\boldsymbol{x}^T) \in \mathbb{R}^n \times \mathbb{S}^n : \boldsymbol{x} \in F\}$.

$\Updownarrow$ "min-max = max min" in the lifting space

concave maxization

$$\gamma^* = \max_{(\boldsymbol{x}, \boldsymbol{W}) \in \widehat{C}} \boldsymbol{M} \bullet \boldsymbol{W} - \boldsymbol{x}^T \boldsymbol{M} \boldsymbol{x}.$$

$\boldsymbol{M} \bullet \boldsymbol{W}$ : the inner product of $\boldsymbol{M}$ and $\boldsymbol{W}$, *i.e.* $\sum_{i,j} M_{ij} W_{ij}$.

$\Downarrow$

Relax the intractable $\widehat{C}$ by a tractable convex $C$;

$$\widehat{C} \subset C \subset \left\{ (\boldsymbol{x}, \boldsymbol{W}) \in \mathbb{R}^n \times \mathbb{S}^n : \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{W} \end{pmatrix} \succeq \boldsymbol{O} \right\}.$$

SDP-SOCP

$$\hat{\gamma} = \max_{(\boldsymbol{x}, \boldsymbol{W}) \in C} \boldsymbol{M} \bullet \boldsymbol{W} - \boldsymbol{x}^T \boldsymbol{M} \boldsymbol{x}$$

$\Rightarrow \quad \gamma^* \leq \hat{\gamma}$.

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Ellipsoidal set $E(\boldsymbol{c}, \gamma) \equiv \{\boldsymbol{x} \in \mathbb{R}^n : \varphi(\boldsymbol{x}, \boldsymbol{c}) \leq \gamma\}, \ \forall \gamma > 0$ (size).

$F$ described by quadratic inequalities

$$F = \{\boldsymbol{x} \in \mathbb{R}^n : \alpha_k + 2\boldsymbol{b}_k^T \boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{Q}_k \boldsymbol{x} \geq 0 \ (1 \leq k \leq p)\}$$

$$= \left\{ \boldsymbol{x} \in \mathbb{R}^n : \begin{pmatrix} \alpha_k & \boldsymbol{b}_k^T \\ \boldsymbol{b}_k & \boldsymbol{Q}_k \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{x}\boldsymbol{x}^T \end{pmatrix} \geq 0 \ (1 \leq k \leq p) \right\},$$

$\widehat{C}$ = convex hull of $\{(\boldsymbol{x}, \boldsymbol{x}\boldsymbol{x}^T) : \boldsymbol{x} \in F\} \subset C$, where

$$C = \left\{ (\boldsymbol{x}, \boldsymbol{W}) : \begin{array}{l} \begin{pmatrix} \alpha_k & \boldsymbol{b}_k^T \\ \boldsymbol{b}_k & \boldsymbol{Q}_k \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{W} \end{pmatrix} \geq 0 \ (1 \leq k \leq p), \\ \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{W} \end{pmatrix} \succeq \boldsymbol{O} \end{array} \right\},$$

SDP-SOCP

$$\hat{\gamma} = \max_{(\boldsymbol{x}, \boldsymbol{W}) \in C} \boldsymbol{M} \bullet \boldsymbol{W} - \boldsymbol{x}^T \boldsymbol{M} \boldsymbol{x} = \boldsymbol{M} \bullet \widehat{\boldsymbol{W}} - \hat{\boldsymbol{c}}^T \boldsymbol{M} \hat{\boldsymbol{c}}$$

$\Longrightarrow F \subset E(\hat{\boldsymbol{c}}, \hat{\gamma})$. Replace $\geq 0$ by $= 0$ $\Rightarrow$ Next applications!

# Outline

# A Sensor Network Localization Problem with Exact Distance

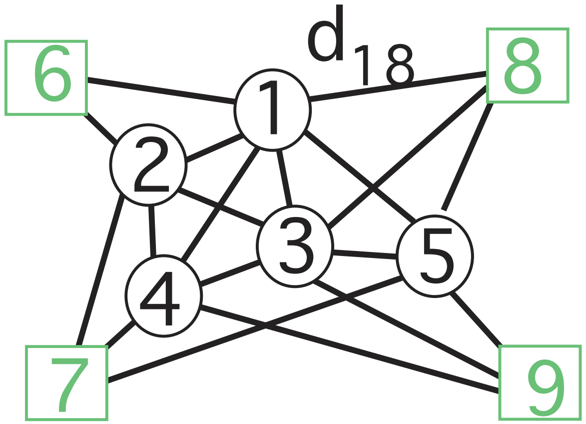
$$\boldsymbol{x}^p \in \mathbb{R}^s \quad : \quad \text{unknown location of sensors } (p = 1, 2, \ldots, m),$$

$$\boldsymbol{x}^r = \boldsymbol{a}^r \in \mathbb{R}^s \quad : \quad \text{known location of anchors } (r = m+1, \ldots, n),$$

$$d_{pq}^2 = \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 \text{ — given for } (p,q) \in \mathcal{N} \qquad (1)$$

$$\mathcal{N} = \{(p,q) : \|\boldsymbol{x}^p - \boldsymbol{x}^q\| \leq \rho = \text{a given radio range}\}$$

$m = 5,\ n = 9$.
$1, \ldots, 5$: sensors
$6, 7, 8, 9$: anchors




Anchors' positions are known.
A distance is given for $\forall$ edge.
Compute locations of sensors.

$\Rightarrow$ Nonconvex QOPs

- SDP relaxation — FSDP by Biswas-Ye '06.
- SFSDP by Kim, Kojima, Waki '09 = a sparse version of FSDP .
- ...

## A Sensor Network Localization Problem with Exact Distance

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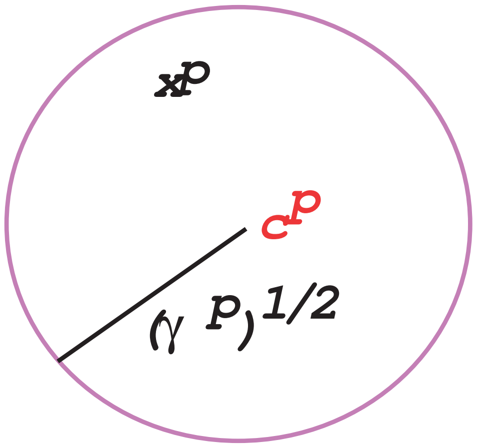
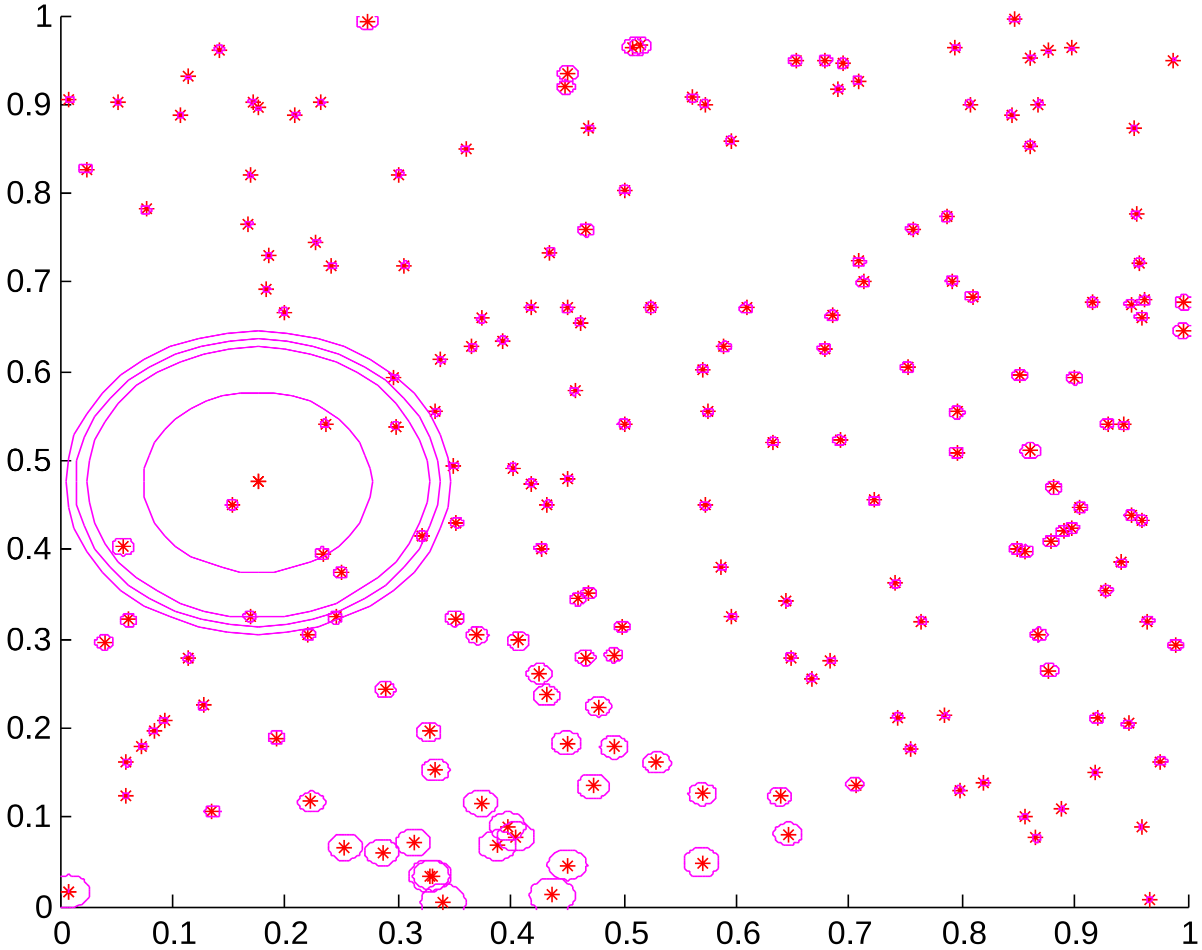
$$d_{pq}^2 = \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 \text{ — given for } (p,q) \in \mathcal{N} \qquad (1)$$
$$\mathcal{N} = \{(p,q) : \|\boldsymbol{x}^p - \boldsymbol{x}^q\| \leq \rho = \text{a given radio range}\}$$

- When $\rho$ is not large enough or $\mathcal{N}$ does not contain enough number of pairs of $p$ and $q$, the system of quadratic equations (1) is underdetermined and/or its SDP relaxation is too weak to locate all sensors uniquely.
- Our method computes $\boldsymbol{c}^p \in \mathbb{R}^s$ and $\gamma^p > 0$ for each sensor $p$ such that the distance from $\boldsymbol{c}^p$ to its unknown location $\boldsymbol{x}^p$ is bounded by $(\gamma^p)^{1/2}$.



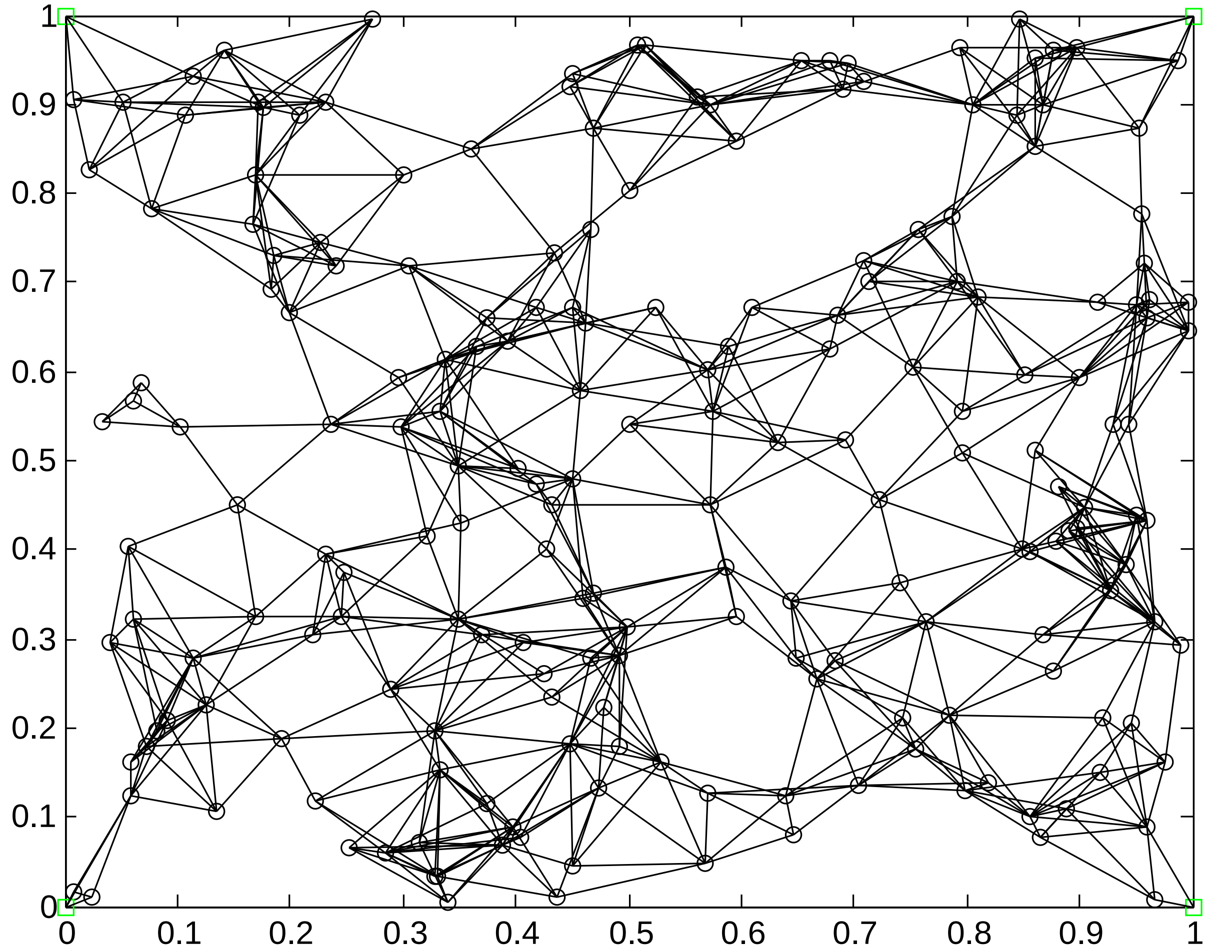
$m = 200$ sensors randomly distributed in $[0,1]^2$, $n - m = 4$ anchors at the corner of $[0,1]^2$, $\rho = 0.14$.




* : $\boldsymbol{c}^p$ = a computed location of censor $p$.

the true location of sensor $p$ is within $(\gamma^p)^{1/2} \leq 0.18$ from $\boldsymbol{c}^p$

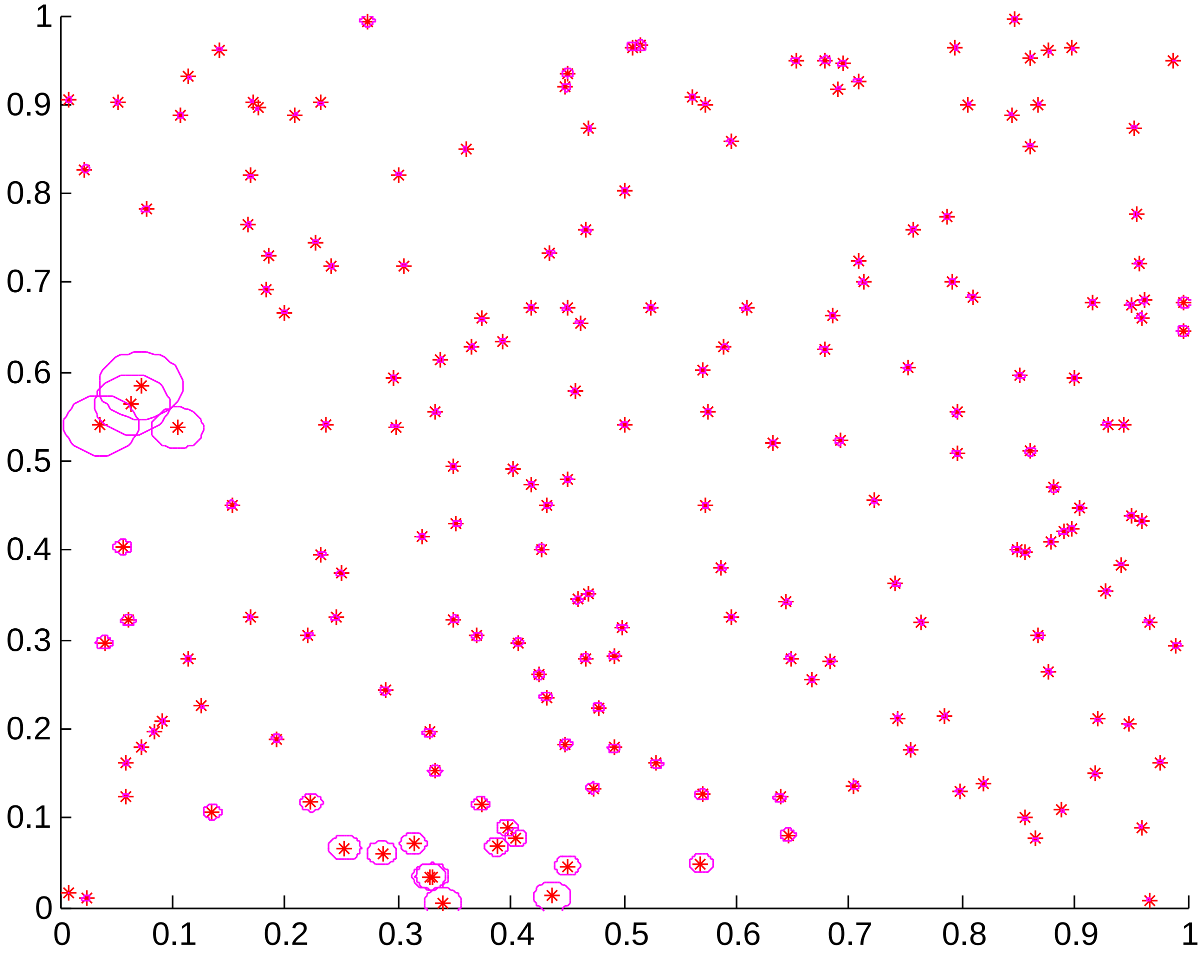
$m = 200$ sensors randomly distributed in $[0, 1]^2$, $n - m = 4$ anchors at the corner of $[0, 1]^2$, $\rho = 0.14$.




the true location $\circ$ of sensor $p$

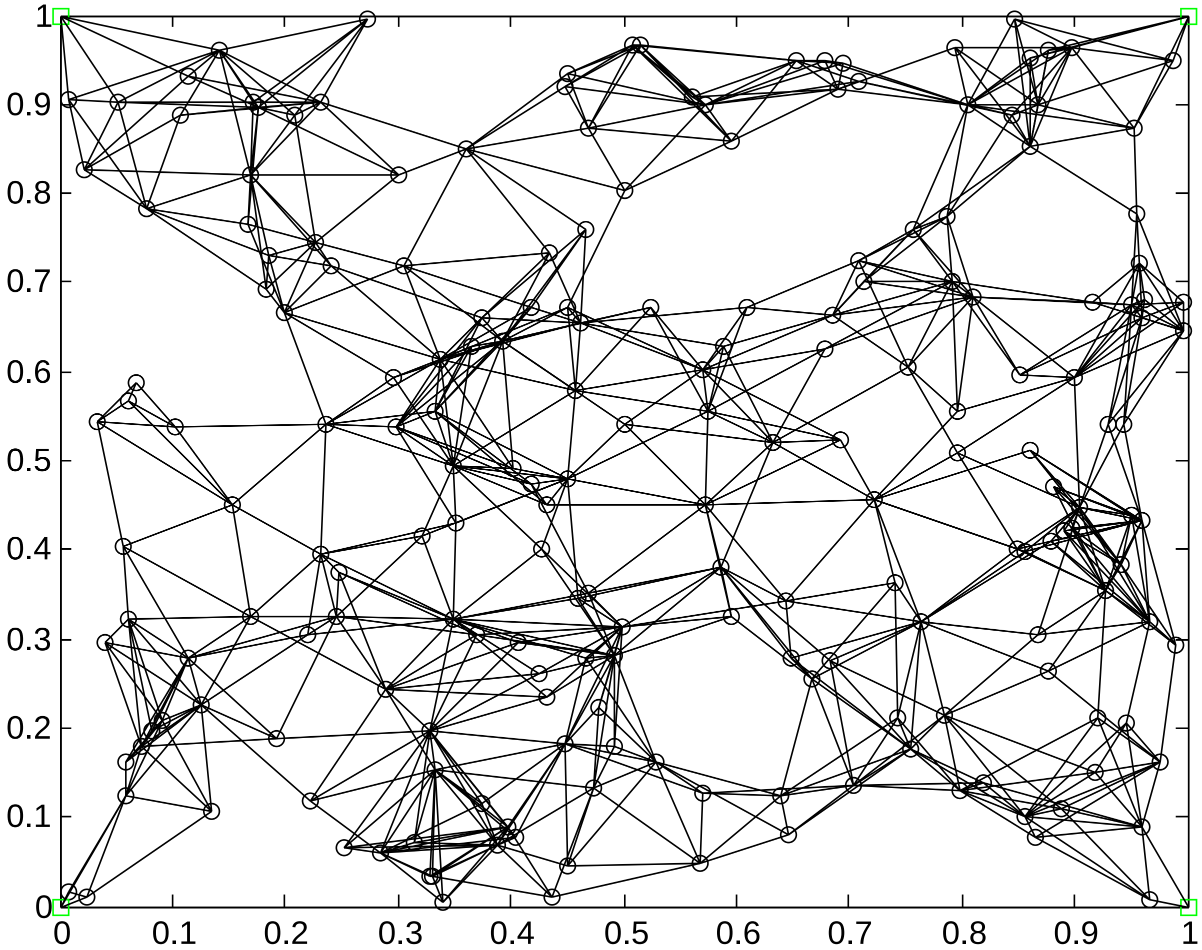
$\circ$——$\circ$ : the edge $(\boldsymbol{x}^p, \boldsymbol{x}^q)$ with a given exact distance

$m = 200$ sensors randomly distributed in $[0,1]^2$, $n - m = 4$ anchors at the corner of $[0,1]^2$, $\rho = 0.15$.




* : $\boldsymbol{c}^p$ = a computed location of censor $p$.
the true location of sensor $p$ is within $(\gamma^p)^{1/2} \leq 0.04$ from $\boldsymbol{c}^p$

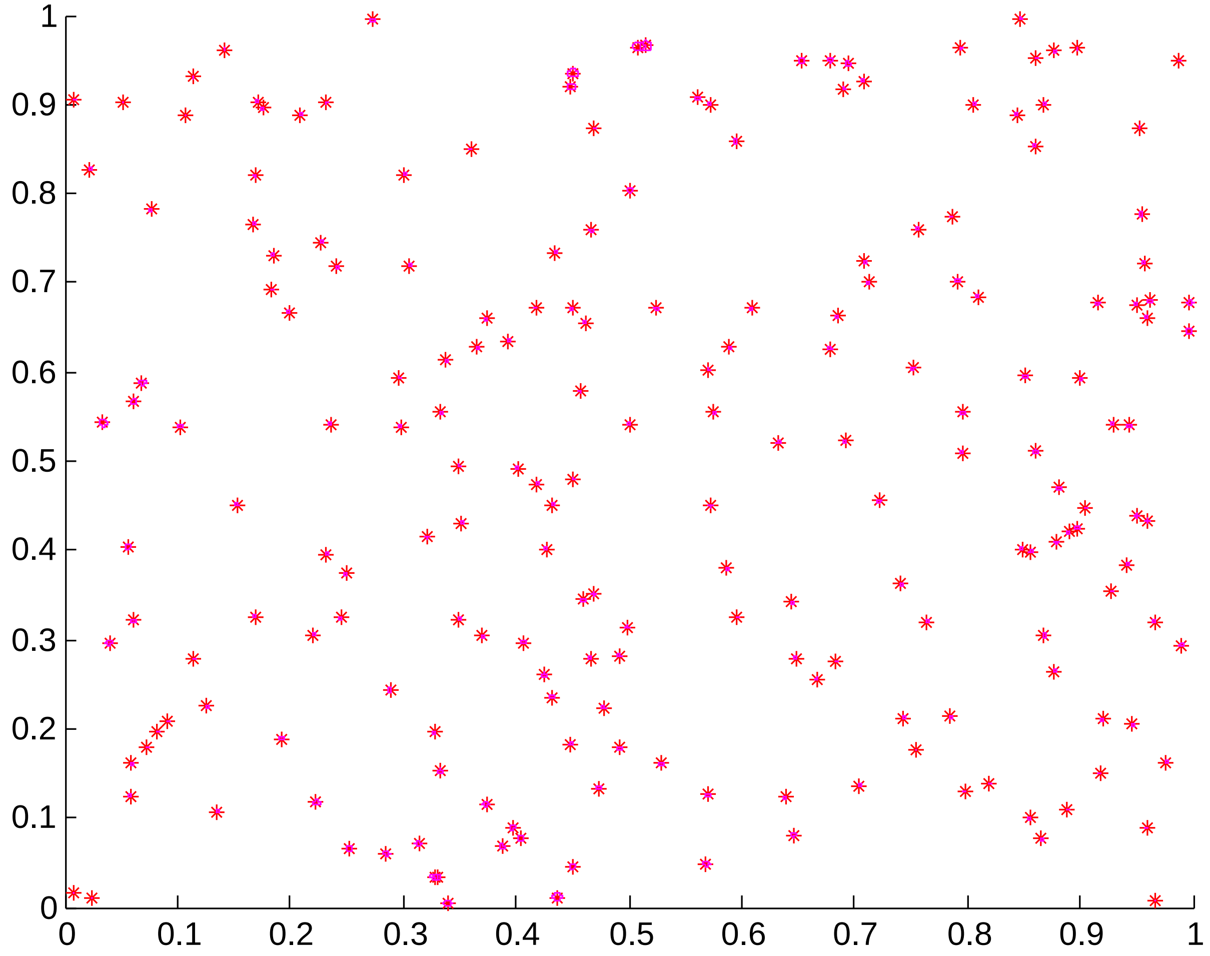
$m = 200$ sensors randomly distributed in $[0,1]^2$, $n - m = 4$ anchors at the corner of $[0,1]^2$, $\rho = 0.15$.




the true location $\circ$ of sensor $p$

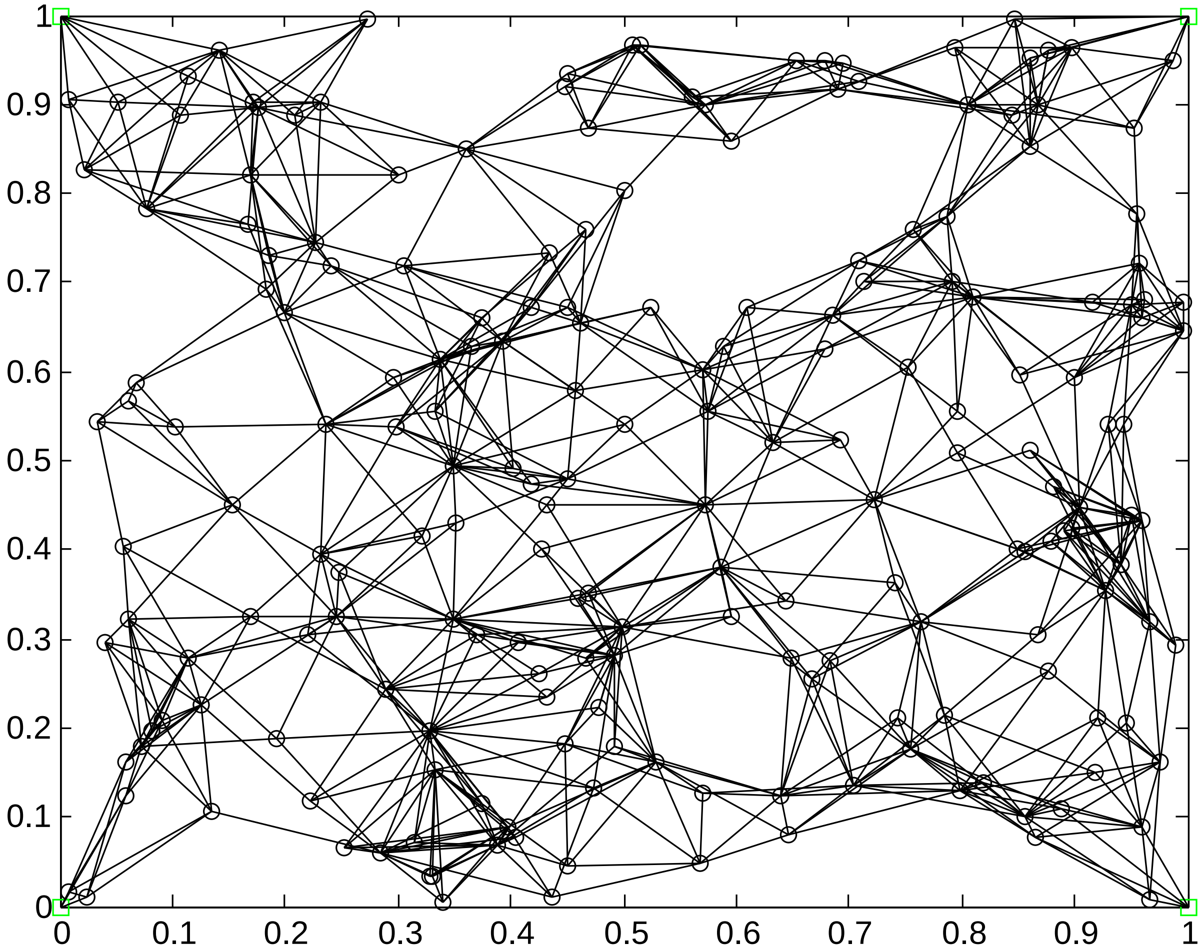
$\circ\!\!-\!\!\!-\!\!\!-\!\!\circ$ : the edge $(\boldsymbol{x}^p, \boldsymbol{x}^q)$ with a given exact distance

$m = 200$ sensors randomly distributed in $[0, 1]^2$, $n - m = 4$ anchors at the corner of $[0, 1]^2$, $\rho = 0.16$.




* : $\boldsymbol{c}^p$ = a computed location of censor $p$.
the true location of sensor $p$ is within $(\gamma^p)^{1/2} \leq 6.0$e-$3$ from $\boldsymbol{c}^p$

$m = 200$ sensors randomly distributed in $[0,1]^2$, $n - m = 4$ anchors at the corner of $[0,1]^2$, $\rho = 0.16$.




the true location $\circ$ of sensor $p$

$\circ\!\!-\!\!\!-\!\!\!-\!\!\circ$ : the edge $(\boldsymbol{x}^p, \boldsymbol{x}^q)$ with a given exact distance

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# Concluding Remarks

We can apply the proposed method to:

- Sensor network localization problems with inexact distance involving measurement error (but the results are not sharp).
- Polynomial optimization problems involving a $0$-$1$ variable $x$ to determine whether $x = 0$ or $x = 1$.
- Polynomial optimization problems involving a pair of variables $x \geq 0$, $y \geq 0$ with complementarity $xy = 0$ to determine whether $x > 0$ or $y > 0$.