Enclosing Ellipsoids and Elliptic Cylinders of Semialgebraic Sets and Their Application to Error Bounds in Polynomial Optimization

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Outline

1. Problem and Some Formulations
2. Theory: Lifting and SDP Relaxation
3. Numerical Results
4. Applications to the Sensor Network Localization Problem
5. Concluding Remarks

Outline

Problem

Given a nonempty compact semialgebraic subset

$$F = \{\boldsymbol{x} \in \mathbb{R}^n : f_k(\boldsymbol{x}) \geq 0 \ (k = 1, 2, \ldots, m)\}$$

of $\mathbb{R}^n$, find a "small" ellipsoid enclosing F. Here $f_k : \mathbb{R}^n \to \mathbb{R}$ denotes a polynomial $(k = 1, 2, \ldots, m)$.

- "small" needs to be specified.

Formulation 1: Minimum volume ellipsoid

F : a nonempty compact semialgebraic subset of $\mathbb{R}^n$.
$\mathcal{E}(\boldsymbol{M}, \boldsymbol{c}) \equiv \{\boldsymbol{x} \in \mathbb{R}^n : (\boldsymbol{x} - \boldsymbol{c})^T \boldsymbol{M}(\boldsymbol{x} - \boldsymbol{c}) \leq 1\}$.

minimize volume of $\mathcal{E}(\boldsymbol{M}, \boldsymbol{c})$

sub.to $F \subset \mathcal{E}(\boldsymbol{M}, \boldsymbol{c}), \ \boldsymbol{M} \succ \boldsymbol{O}, \ \boldsymbol{c} \in \mathbb{R}^n$.

- The most popular in theory
- F consists of a finite number of points $\Rightarrow$ lots of studies $\supset$ (Khachiyan's method 1996)
- Ideal but too difficult in general

Formulation 2: Nie and Demmel 2005

F : a nonempty compact semialgebraic subset of $\mathbb{R}^n$.

$\mathcal{E}(\boldsymbol{P}^{-1}, \boldsymbol{c}) \equiv \{\boldsymbol{x} \in \mathbb{R}^n : (\boldsymbol{x} - \boldsymbol{c})^T \boldsymbol{P}^{-1} (\boldsymbol{x} - \boldsymbol{c}) \leq 1\}$.

$$\begin{array}{ll} \text{minimize} & \text{Trace } \boldsymbol{P} \\ \text{sub.to} & F \subset \mathcal{E}(\boldsymbol{P}^{-1}, \boldsymbol{c}),\ \boldsymbol{P} \succ \boldsymbol{O},\ \boldsymbol{c} \in \mathbb{R}^n. \end{array}$$

$\Longleftarrow$ SOS (Sum Of Squares) relaxation

- A little more general to include parameters.
- Theoretical convergence.
- Still very expensive to apply it to large-scale problems.
  - The SOS relaxation problem becomes a dense problem.

$\Rightarrow$ Less expensive formulation: Fix the shape of the ellipsoid and minimize the size — Ours, next

Our Formulation:

$\boldsymbol{M} \in \mathbb{S}^n_+$ ($n \times n$ positive semidefinite matrices, shape). Define

$\varphi(\boldsymbol{x}, \boldsymbol{c}) \equiv (\boldsymbol{x} - \boldsymbol{c})^T \boldsymbol{M} (\boldsymbol{x} - \boldsymbol{c}), \forall \boldsymbol{x}, \ \forall \boldsymbol{c} \in \mathbb{R}^n$ (center),

Ellipsoidal set $E(\boldsymbol{c}, \gamma) \equiv \{\boldsymbol{x} \in \mathbb{R}^n : \varphi(\boldsymbol{x}, \boldsymbol{c}) \leq \gamma\}, \ \forall \gamma > 0$ (size).

F : a nonempty compact semialgebraic subset of $\mathbb{R}^n$

A min. enclosing ellipsoidal set : $\gamma^* = \min_{\gamma,\ \boldsymbol{c}} \{\gamma : F \subset E(\boldsymbol{c}, \gamma)\}$.

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Application to error bounds in Polynomial Optimization

POP : $f_0^* =$ min $f_0(\boldsymbol{x})$ subject to $f_k(\boldsymbol{x}) \geq 0 \ (k = 1, 2, \ldots, p)$.
Here $f_k : \mathbb{R}^n \to \mathbb{R}$: a polynomial $(k = 0, 1, \ldots, p)$.

Suppose that $\hat{f}_0 \geq f_0^*$ or $\hat{f}_0 = f_0(\hat{\boldsymbol{x}})$ for $\exists$ feasible $\hat{\boldsymbol{x}}$. Let

$$F = \{\boldsymbol{x} \in \mathbb{R}^n : f_k(\boldsymbol{x}) \geq 0, \ (k = 1, 2, \ldots, p), \ f_0(\boldsymbol{x}) \leq \hat{f}_0\}$$

$F \subset E(\boldsymbol{c}, \gamma) \Longrightarrow E(\boldsymbol{c}, \gamma)$ contains all opt. solutions of POP.

$\boldsymbol{M} = \boldsymbol{I} \ \Rightarrow \ \|\boldsymbol{x} - \boldsymbol{c}\| \leq \sqrt{\gamma}$ for $\forall$ opt. sol. $\boldsymbol{x}$
$\boldsymbol{M} =$ diag$(1, 0, \ldots, 0) \ \Rightarrow \ |x_1 - c_1| \leq \sqrt{\gamma}$ for $\forall$ opt. sol. $\boldsymbol{x}$

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- This method can be combined with the SDP relaxation (Lasserre '01) and its sparse variant (Waki et al. '06).

Outline

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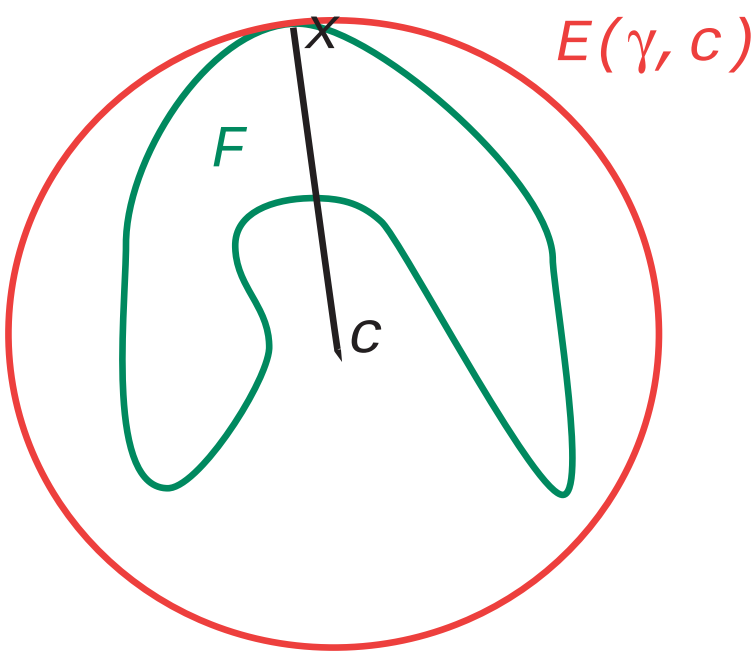
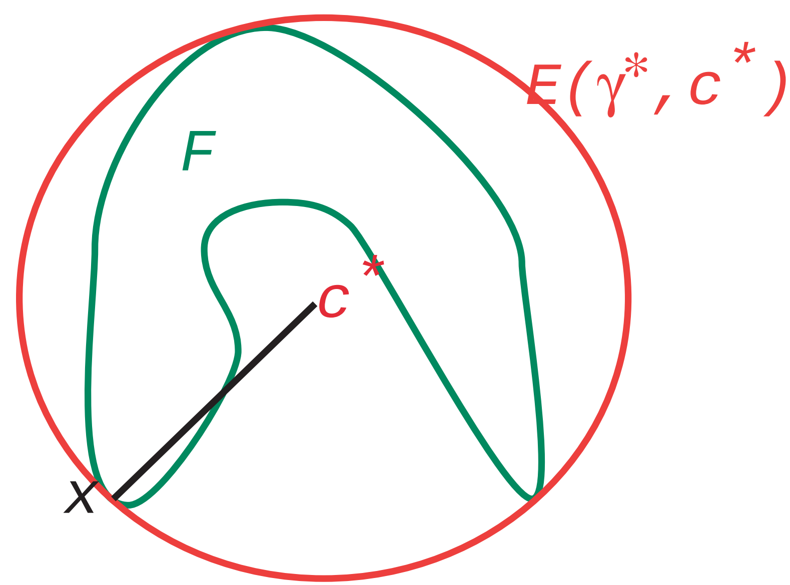
min-max formulation

$$\gamma^* = \min_{\boldsymbol{c}\in\mathbb{R}^n} \max_{\boldsymbol{x}\in F} \varphi(\boldsymbol{x},\boldsymbol{c}) = \max_{\boldsymbol{x}\in F} \varphi(\boldsymbol{x},\boldsymbol{c}^*).$$

Suppose that $\boldsymbol{M}$ = the 2×2 identity matrix





- $\varphi(\boldsymbol{x},\boldsymbol{c}) = \boldsymbol{M} \bullet \boldsymbol{x}\boldsymbol{x}^T - 2\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{c} + \boldsymbol{c}^T\boldsymbol{M}\boldsymbol{c}, \forall \boldsymbol{x}, \forall \boldsymbol{c} \in \mathbb{R}^n$.

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Lifting $\Rightarrow$

$\boldsymbol{M} \in \mathbb{S}^n_+$ ($n \times n$ positive semidefinite matrices, shape),
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Lifting ⇒

Define $\psi(\boldsymbol{x},\boldsymbol{W},\boldsymbol{c}) \equiv \boldsymbol{M} \bullet \boldsymbol{W} - 2\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{c} + \boldsymbol{c}^T\boldsymbol{M}\boldsymbol{c}$,
$C^* \equiv$ the convex hull of $\{(\boldsymbol{x},\boldsymbol{x}\boldsymbol{x}^T) \in \mathbb{R}^n \times \mathbb{S}^n : \boldsymbol{x} \in F\}$.

convex-linear min-max formulation

$$\gamma^* = \min_{\boldsymbol{c}\in\mathbb{R}^n} \left(\max_{(\boldsymbol{x},\boldsymbol{W})\in C^*} \psi(\boldsymbol{x},\boldsymbol{W},\boldsymbol{c}) \right).$$

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$$\gamma^* = \min_{\boldsymbol{c} \in \mathbb{R}^n} \left(\max_{(\boldsymbol{x}, \boldsymbol{W}) \in C^*} \psi(\boldsymbol{x}, \boldsymbol{W}, \boldsymbol{c}) \right).$$
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linear-convex max-min problem
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linear-convex max-min problem

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$\min_{\boldsymbol{c} \in \mathbb{R}^n} \boldsymbol{M} \bullet \boldsymbol{W} - 2\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{c} + \boldsymbol{c}^T\boldsymbol{M}\boldsymbol{c} \quad \Updownarrow \quad \boldsymbol{c}^* = \boldsymbol{x}$: a minimizer

concave maxization

$$\gamma^* = \max_{(\boldsymbol{x}, \boldsymbol{W}) \in C^*} \boldsymbol{M} \bullet \boldsymbol{W} - \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x}.$$

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$\Downarrow$

- Relax the intractable C^* by a tractable convex $\widehat{C}$;

$$L \equiv \left\{ (\boldsymbol{x}, \boldsymbol{W}) \in \mathbb{R}^n \times \mathbb{S}^n : \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{W} \end{pmatrix} \succeq \boldsymbol{O} \right\} \supset \widehat{C} \supset C^*.$$

- Describe $\widehat{C}$ in terms of LMIs.

$\boldsymbol{M} \in \mathbb{S}^n_+$ ($n \times n$ positive semidefinite matrices, shape),
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concave maxization $\quad \gamma^* = \max_{(\boldsymbol{x}, \boldsymbol{W}) \in C^*} \boldsymbol{M} \bullet \boldsymbol{W} - \boldsymbol{x}^T \boldsymbol{M}\boldsymbol{x}.$

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- Describe $\widehat{C}$ in terms of LMIs.

SDP-SOCP $\quad \hat{\gamma} = \max_{(\boldsymbol{x}, \boldsymbol{W}) \in \widehat{C}} \boldsymbol{M} \bullet \boldsymbol{W} - \boldsymbol{x}^T \boldsymbol{M}\boldsymbol{x} \quad \Rightarrow \quad \gamma^* \leq \hat{\gamma}.$

$M \in \mathbb{S}^n_+$ ($n \times n$ positive semidefinite matrices, shape),

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- Under an assumption, $\{C^k$: described in terms of LMIs$\}$;

$$L \supset C^k \supset C^{k+1} \supset C^* \text{ and } \cap_k C^k = C^*$$

by using Lasserre's hierarchy of LMI relaxation '01.

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$$\gamma^* = \max_{(\boldsymbol{x}, \boldsymbol{W}) \in C^*} \boldsymbol{M} \bullet \boldsymbol{W} - \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x}.$$

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$$\hat{\gamma} = \max_{(\boldsymbol{x}, \boldsymbol{W}) \in \widehat{C}} \boldsymbol{M} \bullet \boldsymbol{W} - \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x} \quad \Rightarrow \quad \gamma^* \leq \hat{\gamma}.$$

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concave maxization $\gamma^* = \max_{(\boldsymbol{x}, \boldsymbol{W}) \in C^*} \boldsymbol{M} \bullet \boldsymbol{W} - \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x}.$

Here $C^* \equiv$ the convex hull of $\{(\boldsymbol{x}, \boldsymbol{x}\boldsymbol{x}^T) \in \mathbb{R}^n \times \mathbb{S}^n : \boldsymbol{x} \in F\}$.

$\Downarrow$

- Relax the intractable C^* by a tractable convex $\widehat{C}$;

$$L \equiv \left\{ (\boldsymbol{x}, \boldsymbol{W}) \in \mathbb{R}^n \times \mathbb{S}^n : \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{W} \end{pmatrix} \succeq \boldsymbol{O} \right\} \supset \widehat{C} \supset C^*.$$

- Describe $\widehat{C}$ in terms of LMIs.

SDP-SOCP $\hat{\gamma} = \max_{(\boldsymbol{x}, \boldsymbol{W}) \in \widehat{C}} \boldsymbol{M} \bullet \boldsymbol{W} - \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x}$ $\Rightarrow$ $\gamma^* \leq \hat{\gamma}$.

- When $\widehat{C}$ is described in terms of sparse LMIs, take $\boldsymbol{M}$ which fits their sparsity.

  $\Rightarrow$ a sparse SDP-SOCP which we can solve efficiently.

$\boldsymbol{M} \in \mathbb{S}^n_+$ ($n \times n$ positive semidefinite matrices, shape),
$\varphi(\boldsymbol{x}, \boldsymbol{c}) = \boldsymbol{M} \bullet \boldsymbol{x}\boldsymbol{x}^T - 2\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{c} + \boldsymbol{c}^T\boldsymbol{M}\boldsymbol{c}, \forall \boldsymbol{x}, \forall \boldsymbol{c} \in \mathbb{R}^n$ (center),
Ellipsoidal set $E(\boldsymbol{c}, \gamma) \equiv \{\boldsymbol{x} \in \mathbb{R}^n : \varphi(\boldsymbol{x}, \boldsymbol{c}) \leq \gamma\}, \ \forall \gamma > 0$ (size).

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QOP case

$$F = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \alpha_k + 2\boldsymbol{b}_k^T\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{Q}_k\boldsymbol{x} \geq 0 \ (1 \leq k \leq p) \right\}$$
$$= \left\{ \boldsymbol{x} \in \mathbb{R}^n : \begin{pmatrix} \alpha_k & \boldsymbol{b}_k^T \\ \boldsymbol{b}_k & \boldsymbol{Q}_k \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{x}\boldsymbol{x}^T \end{pmatrix} \geq 0 \ (1 \leq k \leq p) \right\},$$

$\boldsymbol{M} \in \mathbb{S}^n_+$ ($n \times n$ positive semidefinite matrices, shape),
$\varphi(\boldsymbol{x}, \boldsymbol{c}) = \boldsymbol{M} \bullet \boldsymbol{x}\boldsymbol{x}^T - 2\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{c} + \boldsymbol{c}^T\boldsymbol{M}\boldsymbol{c}, \forall \boldsymbol{x}, \forall \boldsymbol{c} \in \mathbb{R}^n$ (center),
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$$= \left\{\boldsymbol{x} \in \mathbb{R}^n : \begin{pmatrix} \alpha_k & \boldsymbol{b}_k^T \\ \boldsymbol{b}_k & \boldsymbol{Q}_k \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{x}\boldsymbol{x}^T \end{pmatrix} \geq 0 \ (1 \leq k \leq p)\right\},$$

Let

$$\widehat{C} = \left\{ (\boldsymbol{x}, \boldsymbol{W}) : \begin{array}{l} \begin{pmatrix} \alpha_k & \boldsymbol{b}_k^T \\ \boldsymbol{b}_k & \boldsymbol{Q}_k \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{W} \end{pmatrix} \geq 0 \ (1 \leq k \leq p), \\ \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{W} \end{pmatrix} \succeq \boldsymbol{O} \end{array} \right\},$$

SDP-SOCP

$$\hat{\gamma} = \max_{(\boldsymbol{x}, \boldsymbol{W}) \in \widehat{C}} \boldsymbol{M} \bullet \boldsymbol{W} - \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x} = \boldsymbol{M} \bullet \widehat{\boldsymbol{W}} - \hat{\boldsymbol{c}}^T\boldsymbol{M}\hat{\boldsymbol{c}}$$

$$\Longrightarrow F \subset E(\hat{\boldsymbol{c}}, \hat{\gamma}).$$

Outline

Outline

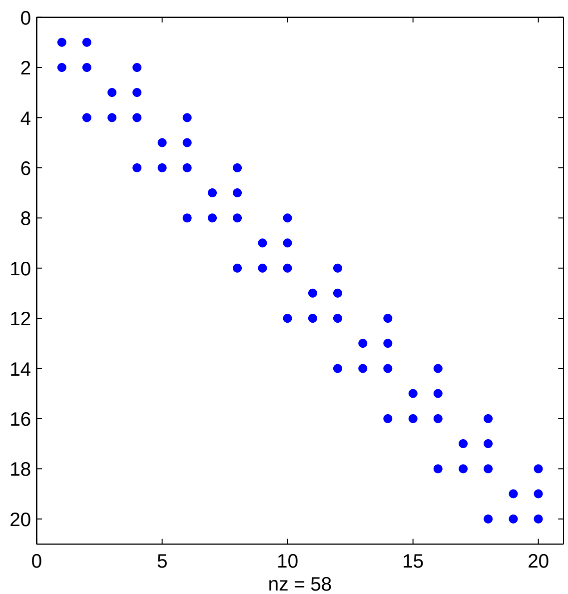
- SparsePOP (Waki et al. '08) for constructing sparse SDP relaxation problems of POPs.
- SeDuMi1.21 (Sturm, Polik '09) for solving SDP relaxation problems to compute an approx. opt. sol. of POPs and for solving SDP-SOCPs to compute error bounds.
- MATLAB Optimization Toolbox to refine the approx. opt. sol. obtained by SeDuMi for constrained optimization problems.
- 2.8GHz Intel Xeon with 4GB Memory.

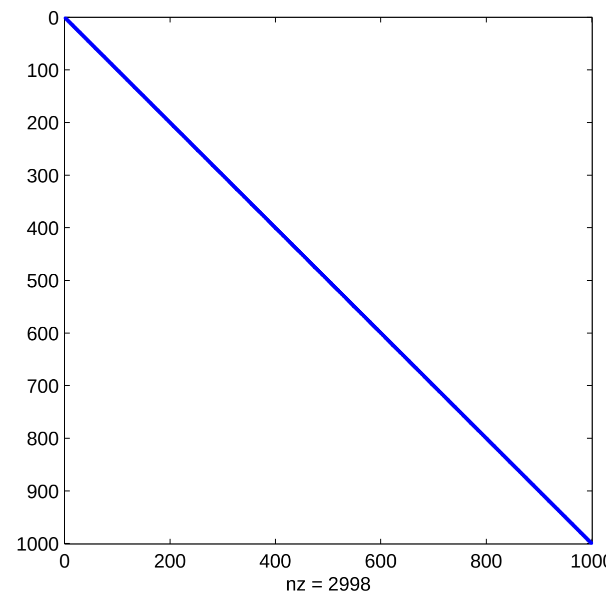
Unconstrained min. of ChainedWood function $f_C(\boldsymbol{x})$

$$f_C(\boldsymbol{x}) = 1 + \sum_{i \in J} \left(100(x_{i+1} - x_i^2)^2 + (1 - x_i)^2 + 90(x_{i+3} - x_{i+2}^2)^2 + (1 - x_{i+2})^2 + 10(x_{i+1} + x_{i+3} - 2)^2 + 0.1(x_{i+1} - x_{i+3})^2\right)$$

Here $J = \{1, 3, 5, \ldots, n-3\}$ and n is a multiple of 4.

Sparsity pattern of the Hessian matrix





$n = 20$, 20×20 matrix

no. of nonzeros = $\frac{58}{400}$

$n = 1000$, 1000×1000 matrix

no. of nonzeros = $\frac{2,988}{1,000,000}$

- Sparse enough to solve larger scale problems.

Unconstrained min. of ChainedWood function $f_C(\boldsymbol{x})$

$$f_C(\boldsymbol{x}) = 1 + \sum_{i \in J} \Big(100(x_{i+1} - x_i^2)^2 + (1 - x_i)^2 + 90(x_{i+3} - x_{i+2}^2)^2 + (1 - x_{i+2})^2 + 10(x_{i+1} + x_{i+3} - 2)^2 + 0.1(x_{i+1} - x_{i+3})^2\Big)$$

Here $J = \{1, 3, 5, \ldots, n-3\}$ and n is a multiple of 4.

Unconstrained min. of ChainedWood function $f_C(\boldsymbol{x})$

$$f_C(\boldsymbol{x}) = 1 + \sum_{i \in J} \left(100(x_{i+1} - x_i^2)^2 + (1 - x_i)^2 + 90(x_{i+3} - x_{i+2}^2)^2 + (1 - x_{i+2})^2 + 10(x_{i+1} + x_{i+3} - 2)^2 + 0.1(x_{i+1} - x_{i+3})^2\right)$$

Here $J = \{1, 3, 5, \ldots, n-3\}$ and n is a multiple of 4.

$\boldsymbol{M}$ = the $n \times n$ identity matrix.

n	RelObjErr at $\hat{\boldsymbol{x}}$	E.Time for $\hat{\boldsymbol{x}}$	Error bound E.time	Error bound $\sqrt{\hat{\gamma}}/\lVert\hat{\boldsymbol{c}}\rVert$
1000	4.4e-4	2.4	4.7	4.9e-3
2000	8.8e-4	5.7	11.6	4.9e-3
4000	1.8e-3	14.6	30.3	1.5e-3

$\hat{\boldsymbol{x}}$ = an approx. optimal solution,

$$\text{RelObjErr} = \frac{|\text{lbd. for opt. val.} - f_C(\hat{\boldsymbol{x}})|}{|f_C(\hat{\boldsymbol{x}})|}$$

$\lVert\boldsymbol{x} - \hat{\boldsymbol{c}}\rVert / \lVert\hat{\boldsymbol{c}}\rVert \leq \sqrt{\hat{\gamma}}/\lVert\hat{\boldsymbol{c}}\rVert$, $\forall$ global minimizer $\boldsymbol{x}$

alkyl.gms from globallib

$$\begin{aligned}
&\text{min} \quad -6.3x_5x_8 + 5.04x_2 + 0.35x_3 + x_4 + 3.36x_6 \\
&\text{sub.to} \quad x_5x_{12} - x_2(1.12 + 0.132x_9 - 0.0067x_9^2) = 0, \\
&x_8x_{13} - 0.01x_9(1.098 - 0.038x_9) - 0.325x_7 = 0.574, \\
&0.98x_4 - x_7(0.01x_5x_{10} + x_4) = 0, \ -x_2x_9 + 10x_3 + x_6 = 0, \\
&-0.820x_2 + x_5 - 0.820x_6 = 0, \ x_1x_{11} - 3x_8 = -1.33, \\
&x_{10}x_{14} + 22.2x_{11} = 35.82, \ \text{lbd}_i \leq x_i \leq \text{ubd}_i \ (i = 1, 2, \ldots, 14).
\end{aligned}$$

$$\frac{|\text{lbd for opt.val.} - \text{approx. opt.val } f_0(\hat{\boldsymbol{x}})|}{|\text{approx. opt.val } f_0(\hat{\boldsymbol{x}})|} = 6.7e\text{-}6$$

max error in equalities at $\hat{\boldsymbol{x}} = 5.2e\text{-}9$

$F = \{\boldsymbol{x} \in \mathbb{R}^{14} : \text{ feasible and } f_0(\boldsymbol{x}) \leq f_0(\hat{\boldsymbol{x}})\} \subset E(\hat{\boldsymbol{c}}, \hat{\gamma})$

alkyl.gms from globallib

$$\begin{aligned}
&\text{min} \quad -6.3x_5x_8 + 5.04x_2 + 0.35x_3 + x_4 + 3.36x_6 \\
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max error in equalities at $\hat{\boldsymbol{x}} = 5.2e$-9

$$F = \{\boldsymbol{x} \in \mathbb{R}^{14} : \text{ feasible and } f_0(\boldsymbol{x}) \leq f_0(\hat{\boldsymbol{x}})\} \subset E(\hat{\boldsymbol{c}}, \hat{\gamma})$$

$\boldsymbol{M} = \boldsymbol{I} \in \mathbb{S}^{14} \Rightarrow \hat{\boldsymbol{c}} = (1.7037030, 1.5847109, \ldots)$, $\sqrt{\hat{\gamma}} = 1.6$e-4.

$\|\boldsymbol{x} - \hat{\boldsymbol{c}}\| \leq \sqrt{\hat{\gamma}}$ for $\forall$ opt. sol. $\boldsymbol{x} \in \mathbb{R}^{14}$.

alkyl.gms from globallib

$$
\begin{array}{ll}
\min & -6.3x_5x_8 + 5.04x_2 + 0.35x_3 + x_4 + 3.36x_6 \\
\text{sub.to} & x_5x_{12} - x_2(1.12 + 0.132x_9 - 0.0067x_9^2) = 0,
\end{array}
$$

$$
x_8x_{13} - 0.01x_9(1.098 - 0.038x_9) - 0.325x_7 = 0.574,
$$

$$
0.98x_4 - x_7(0.01x_5x_{10} + x_4) = 0,\ -x_2x_9 + 10x_3 + x_6 = 0,
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$$
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$$

$$
x_{10}x_{14} + 22.2x_{11} = 35.82,\ \text{lbd}_i \leq x_i \leq \text{ubd}_i\ (i = 1, 2, \ldots, 14).
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$$
\frac{|\text{lbd for opt.val.} - \text{approx. opt.val } f_0(\hat{\boldsymbol{x}})|}{|\text{approx. opt.val } f_0(\hat{\boldsymbol{x}})|} = 6.7e\text{-}6
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$F = \{\boldsymbol{x} \in \mathbb{R}^{14} :$ feasible and $f_0(\boldsymbol{x}) \leq f_0(\hat{\boldsymbol{x}})\} \subset E(\hat{\boldsymbol{c}}, \hat{\gamma})$

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$\|\boldsymbol{x} - \hat{\boldsymbol{c}}\| \leq \sqrt{\hat{\gamma}}$ for $\forall$ opt. sol. $\boldsymbol{x} \in \mathbb{R}^{14}$.

$\boldsymbol{M} =$ diag$(1, 0, \ldots, 0) \in \mathbb{S}^{14} \Rightarrow \hat{c}_1 = 1.7037017$, $\sqrt{\hat{\gamma}} = 1.0$e-5.

$|x_1 - \hat{c}_1| \leq \sqrt{\hat{\gamma}}$ for $\forall$ opt. sol. $\boldsymbol{x} \in \mathbb{R}^{14}$.

Nonconvex QPs from globalib

$\boldsymbol{M}$ = the $n \times n$ identity matrix

Problem	n	RelObjErr	Error bound $\sqrt{\hat{\gamma}}$	Error bound $\sqrt{\hat{\gamma}}/\|\|\hat{\boldsymbol{c}}\|\|$	E.time sdpa
ex2_1_3	13	1.1e-9	4.9e-4	4.9e-4	0.5
ex2_1_5	10	3.5e-10	4.7e-4	1.7e-4	0.8
ex2_1_8	24	3.5e-9	5.4e-2	1.3e-3	9.5
ex9_1_2†	10	1.8e-2	4.2	0.53	0.2
ex9_1_5†	13	6.2e-2	4.7	1.0	0.3
ex9_2_3	16	2.8e-7	1.4e-2	2.6e-4	0.2

$$\text{RelObjErr} = \frac{|\text{approx. otp. val.} - \text{l. bd. for otp. val.}|}{|\text{approx. otp. val.}|}$$

$$\|\boldsymbol{x} - \hat{\boldsymbol{c}}\| \leq \sqrt{\hat{\gamma}}, \ \forall \text{ global minimizer } \boldsymbol{x}$$

† : multiple solutions

More details on ex9_1_2[†]

min.	$-x_1 - 3x_2$
sub. to	5 linear equations in x_j $(j = 1, 2, \ldots, 10)$,
	$x_3 x_7 = 0,\ x_4 x_8 = 0,\ x_5 x_9 = 0,\ x_6 x_{10} = 0,$
	$0 \leq x_j \leq 5\ (j = 1, 2, \ldots, 10).$

$\boldsymbol{M}$ = diag(the ith unit coordinate vector) $(i = 3, 4, 5, 6, 8, 9)$

$\Rightarrow$ $|x_i - \hat{c}_i| \leq \sqrt{\hat{\gamma}}$ for $\forall$ opt. sol. $\boldsymbol{x}$

i	$\hat{c}_i$	$\sqrt{\hat{\gamma}}$		at $\forall$ opt. sol.
3	2.9995	0.0089	$\Rightarrow$	$x_3 > 0$, $x_7 = 0$
4	0.0002	0.0279		
5	0.0009	0.0148		
6	4.0002	0.0123	$\Rightarrow$	$x_6 > 0$, $x_{10} = 0$
8	1.000	1.0001		
9	3.000	2.0004	$\Rightarrow$	$x_9 > 0$, $x_5 = 0$

Fixing $x_5 = x_7 = x_{10} = 0$, we obtain the reduced problem $\Rightarrow$

Reduced ex9_1_2† with fixing $x_5 = x_7 = x_{10} = 0$

min.	$-x_1 - 3x_2$
sub. to	5 linear equations in x_j $(j = 1, 2, 3, 4, 6, 8, 9)$,
	$x_4 x_8 = 0, \ 0 \le x_j \le 5 \ (j = 1, 2, 3, 4, 6, 8, 9)$.

$\boldsymbol{M}$ = diag(the ith unit coordinate vector) $(i = 1, 2, 3, 4, 6, 8, 9)$

$\Rightarrow |x_i - \hat{c}_i| \le \sqrt{\hat{\gamma}}$ for $\forall$ opt. sol. $\boldsymbol{x}$

i	$\hat{c}_i$	$\sqrt{\hat{\gamma}}$	i	$\hat{c}_i$	$\sqrt{\rho^*}$
1	4.0000	0.0002	2	4.0000	0.0002
3	3.0000	0.0006	4	0.0000	0.0006
6	4.0000	0.0004			
8	1.0000	1.0000	$\Rightarrow 0.0000 \le x_8 \le 2.0000$		
9	3.0000	2.0000	$\Rightarrow 1.0000 \le x_9 \le 5.0000$		

We can verify that the optimal solutions are:

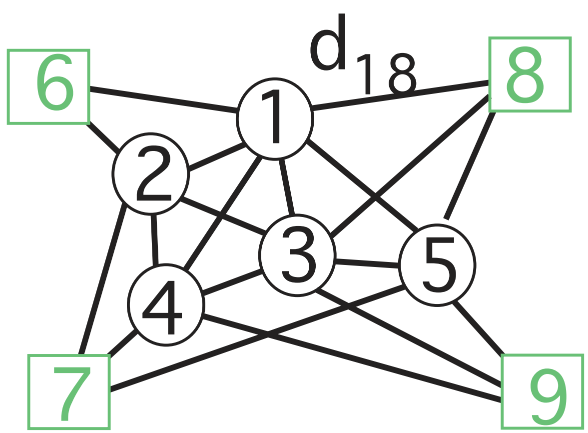
$$x_1 = x_2 = x_6 = 4, \ x_3 = 3, \ x_4 = 0,$$

$$0 \le x_8 = (x_9 - 1)/2 \le 2, \ 1 \le x_9 \le 5.$$

Outline

A Sensor Network Localization Problem with Exact Distance

$m = 5,\ n = 9$.
$1, \ldots, 5$: sensors
$6, 7, 8, 9$: anchors



- Sensors' locations are unknown.
- Anchors' locatios are known.
- A distance is given for $\forall$ edge.

Compute locations of sensors.
$\Rightarrow$ Nonconvex QOPs

A Sensor Network Localization Problem with Exact Distance

$$\boldsymbol{x}^p \in \mathbb{R}^2 \quad : \quad \text{unknown location of sensors } (p = 1, 2, \ldots, m),$$

$$\boldsymbol{x}^r = \boldsymbol{a}^r \in \mathbb{R}^2 \quad : \quad \text{known location of anchors } (r = m+1, \ldots, n),$$

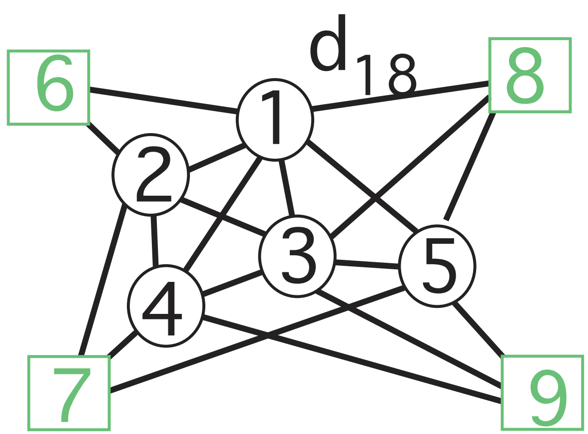
$$d_{pq}^2 = \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 \text{ — given for } (p, q) \in \mathcal{E} \qquad (1)$$

$$\mathcal{E} = \{(p, q) : \|\boldsymbol{x}^p - \boldsymbol{x}^q\| \leq \rho = \text{a given radio range}\}$$

$m = 5, \ n = 9.$

$1, \ldots, 5$: sensors

$6, 7, 8, 9$: anchors



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$$\mathcal{E} \quad = \quad \{(p, q) : \|\boldsymbol{x}^p - \boldsymbol{x}^q\| \leq \rho = \text{a given radio range}\}$$

- FSDP by Biswas-Ye '06, SDP relaxation of (1)
  — Powerful in theory;
  FSDP computes exact locations $\boldsymbol{x}^p$ $(p = 1, 2, \ldots, m)$ if
  "(1) is uniquely localizable"
  = "the rigidity of $G(\{1, 2, \ldots, m\}, \mathcal{E})$ + a certain condition".
  But expensive in computation.
- SFSDP by Kim, Kojima, Waki '09 = a sparse version of FSDP which is as effective as FSDP in theory but is more efficient in computation.

A Sensor Network Localization Problem with Exact Distance

$$
\begin{aligned}
\boldsymbol{x}^p \in \mathbb{R}^2 &\quad : \quad \text{unknown location of sensors } (p = 1, 2, \ldots, m), \\
\boldsymbol{x}^r = \boldsymbol{a}^r \in \mathbb{R}^2 &\quad : \quad \text{known location of anchors } (r = m+1, \ldots, n), \\
d_{pq}^2 &= \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 \text{ — given for } (p,q) \in \mathcal{E} \qquad (1) \\
\mathcal{E} &= \{(p,q) : \|\boldsymbol{x}^p - \boldsymbol{x}^q\| \leq \rho = \text{a given radio range}\}
\end{aligned}
$$

A Sensor Network Localization Problem with Exact Distance

$$\boldsymbol{x}^p \in \mathbb{R}^2 \quad : \quad \text{unknown location of sensors } (p = 1, 2, \ldots, m),$$

$$\boldsymbol{x}^r = \boldsymbol{a}^r \in \mathbb{R}^2 \quad : \quad \text{known location of anchors } (r = m+1, \ldots, n),$$

$$d_{pq}^2 = \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 \text{ — given for } (p,q) \in \mathcal{E} \qquad (1)$$

$$\mathcal{E} = \{(p,q) : \|\boldsymbol{x}^p - \boldsymbol{x}^q\| \leq \rho = \text{a given radio range}\}$$

Numerical Results: 20,000 sensors randomly distributed in $[0,1] \times [0,1]$, 4 anchors at the corner and $\rho = 0.1$

σ	RMSD	SDPA E.time
0.0	6.9e-6	182.9
0.1	7.6e-3	403.0
0.2	1.1e-2	402.6

$$\text{RMSD} = \left(\frac{1}{m} \sum_{p=1}^{m} \|\boldsymbol{x}^p - \boldsymbol{a}^p\| \right)^{1/2}.$$

$\boldsymbol{a}^p$: true location of p

$\sigma > 0 \Rightarrow d_{pq} = (1+\xi) \times$ true distance, diferrent formulation:

$$\min \sum_{(p,q)\in\mathcal{E}} \left| \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 - d_{pq}^2 \right| \;\Leftarrow\; \text{sparse SDP relaxation.}$$

Here ξ is chosen from $N(0,\sigma)$.

A Sensor Network Localization Problem with Exact Distance

$$\boldsymbol{x}^p \in \mathbb{R}^2 \quad : \quad \text{unknown location of sensors } (p = 1, 2, \ldots, m),$$

$$\boldsymbol{x}^r = \boldsymbol{a}^r \in \mathbb{R}^2 \quad : \quad \text{known location of anchors } (r = m+1, \ldots, n),$$

$$d_{pq}^2 = \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 \text{ — given for } (p,q) \in \mathcal{E} \qquad (1)$$

$$\mathcal{E} = \{(p,q) : \|\boldsymbol{x}^p - \boldsymbol{x}^q\| \leq \rho = \text{a given radio range}\}$$

A Sensor Network Localization Problem with Exact Distance

$$\begin{aligned} \boldsymbol{x}^p \in \mathbb{R}^2 &\quad : \quad \text{unknown location of sensors } (p = 1, 2, \ldots, m), \\ \boldsymbol{x}^r = \boldsymbol{a}^r \in \mathbb{R}^2 &\quad : \quad \text{known location of anchors } (r = m+1, \ldots, n), \\ d_{pq}^2 &= \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 \text{ — given for } (p,q) \in \mathcal{E} \qquad (1) \\ \mathcal{E} &= \{(p,q) : \|\boldsymbol{x}^p - \boldsymbol{x}^q\| \leq \rho = \text{a given radio range}\} \end{aligned}$$

- Some numerical results of SFSD combined with our method for an ellipoidal set enclosing $F = \{(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^m) : d_{pq}^2 = \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 \text{ for } (p,q) \in \mathcal{E}\}$.

A Sensor Network Localization Problem with Exact Distance

$$\begin{aligned}
\boldsymbol{x}^p \in \mathbb{R}^2 \quad &: \quad \text{unknown location of sensors } (p = 1, 2, \ldots, m), \\
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d_{pq}^2 \quad &= \quad \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 \text{ — given for } (p, q) \in \mathcal{E} \qquad (1) \\
\mathcal{E} \quad &= \quad \{(p, q) : \|\boldsymbol{x}^p - \boldsymbol{x}^q\| \leq \rho = \text{a given radio range}\}
\end{aligned}$$

A Sensor Network Localization Problem with Exact Distance

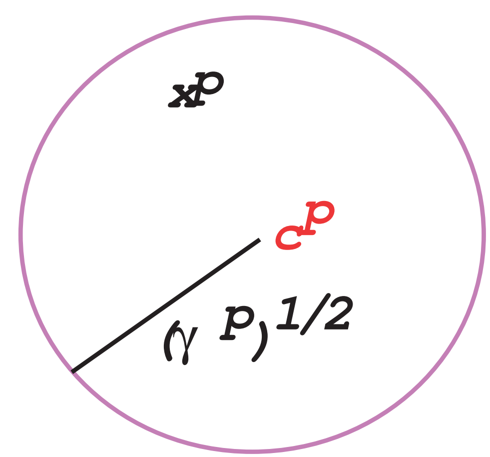
$$\boldsymbol{x}^p \in \mathbb{R}^2 \quad : \quad \text{unknown location of sensors } (p = 1, 2, \ldots, m),$$

$$\boldsymbol{x}^r = \boldsymbol{a}^r \in \mathbb{R}^2 \quad : \quad \text{known location of anchors } (r = m+1, \ldots, n),$$

$$d_{pq}^2 = \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 \text{ — given for } (p, q) \in \mathcal{E} \qquad (1)$$

$$\mathcal{E} = \{(p, q) : \|\boldsymbol{x}^p - \boldsymbol{x}^q\| \leq \rho = \text{a given radio range}\}$$

Problem: For each sensor $p = 1, 2, \ldots, m$, compute $\boldsymbol{c}^p \in \mathbb{R}^2$ and $\gamma^p > 0$ such that the distance from $\boldsymbol{c}^p$ to its unknown location is bounded by $(\gamma^p)^{1/2}$.



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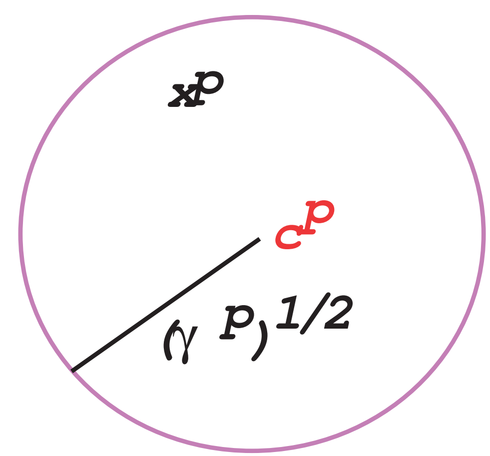
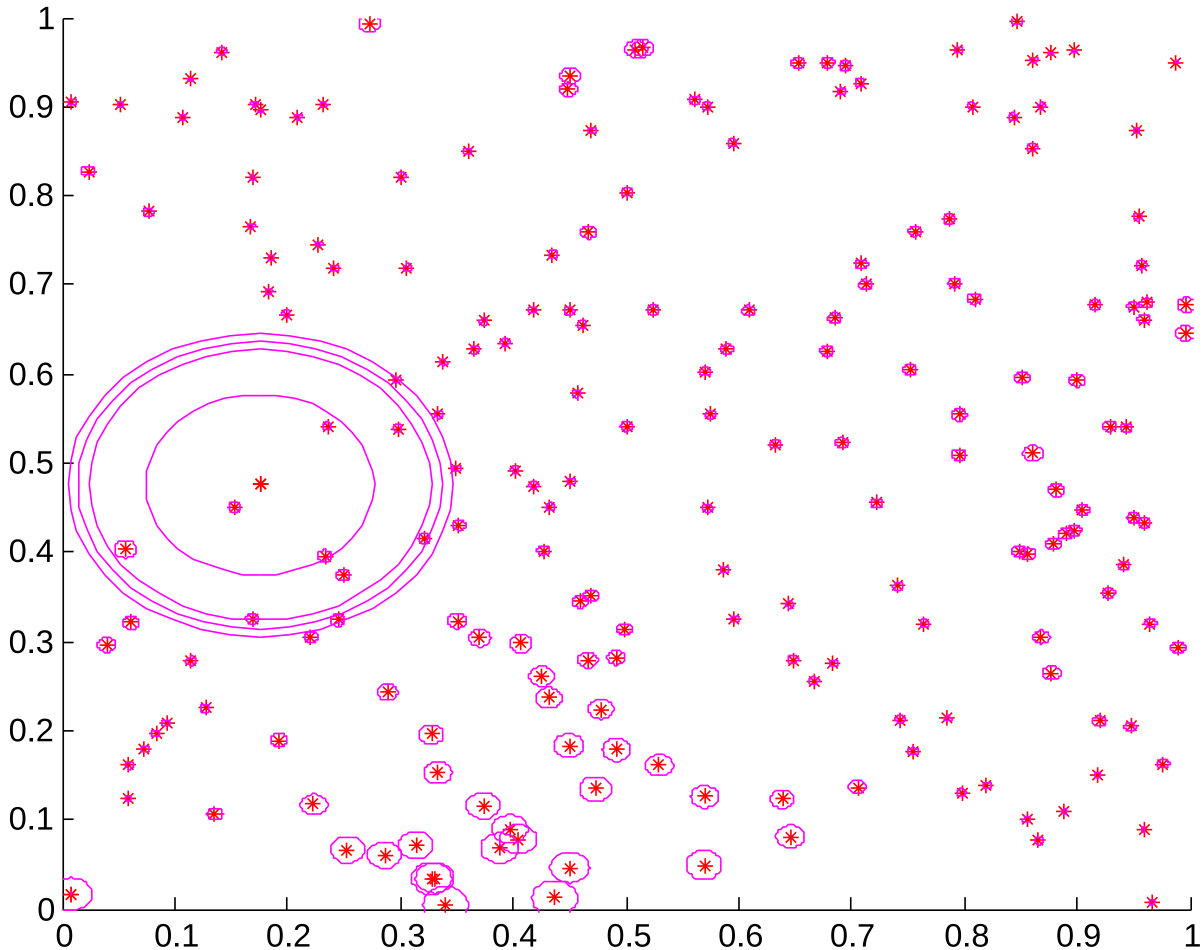
- When ρ is not large enough or $\mathcal{E}$ does not contain enough number of edges, (1) is underdetermined and/or its SDP relaxation is too weak to locate all sensors uniquely.
- Our method + SFSDP computes $\boldsymbol{c}^p \in \mathbb{R}^2$ and $\gamma^p > 0$ for each sensor p such that the distance from $\boldsymbol{c}^p$ to its unknown location $\boldsymbol{x}^p$ is bounded by $(\gamma^p)^{1/2}$.

$\Downarrow$

- If $\gamma^p = 0$ then $\boldsymbol{c}^p$ = the exact location of p (Biswas-Ye ’06).



$m = 200$ sensors randomly distributed in $[0,1]^2$, $n - m = 4$ anchors at the corner of $[0,1]^2$, $\rho = 0.14$.



* : $\boldsymbol{c}^p$ = a computed location of censor p.

the true location $\boldsymbol{x}^p$ of sensor p is within $(\gamma^p)^{1/2} \leq 0.18$ from $\boldsymbol{c}^p$

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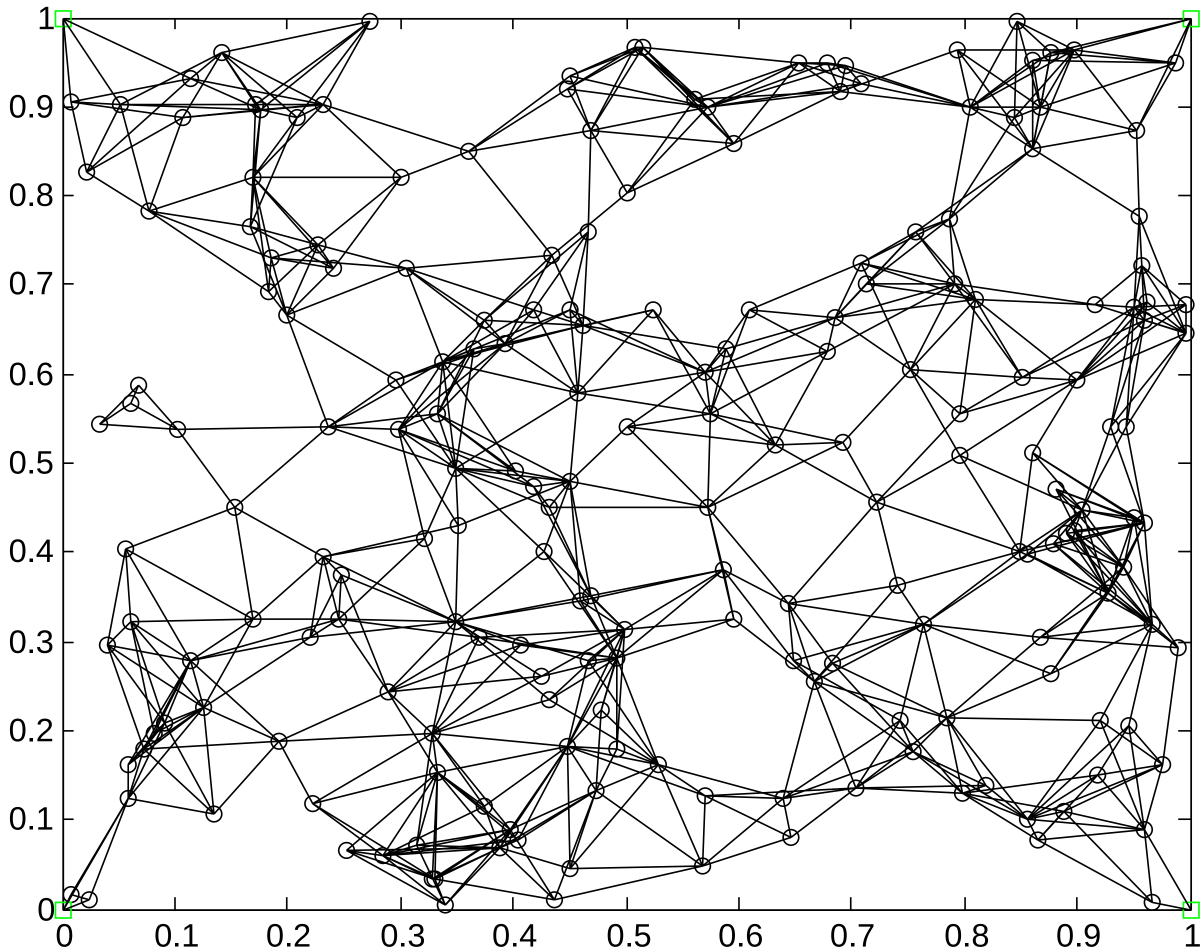
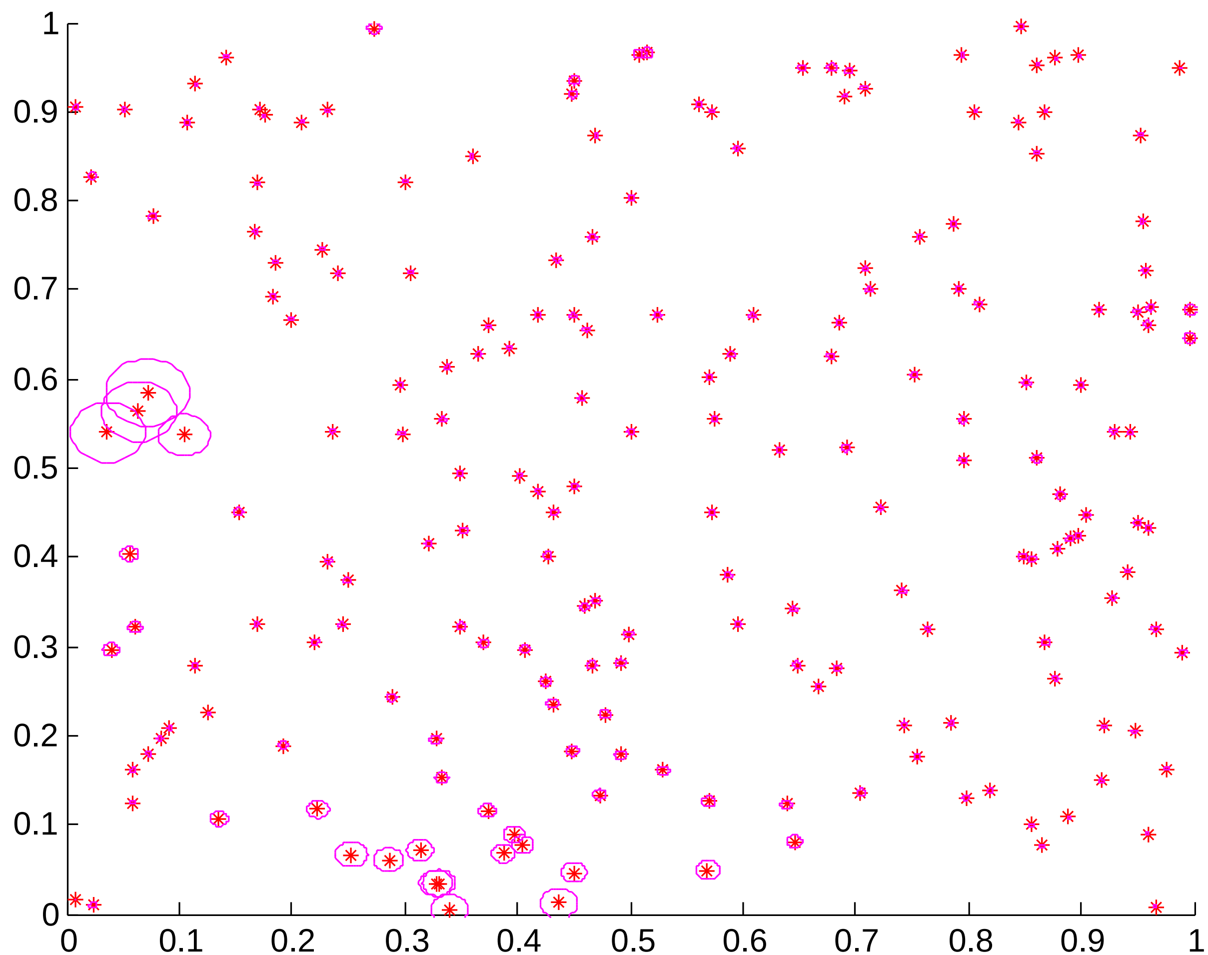


the true location ○ of sensor p

○——○ : the edge $(\boldsymbol{x}^p, \boldsymbol{x}^q)$ with a given exact distance

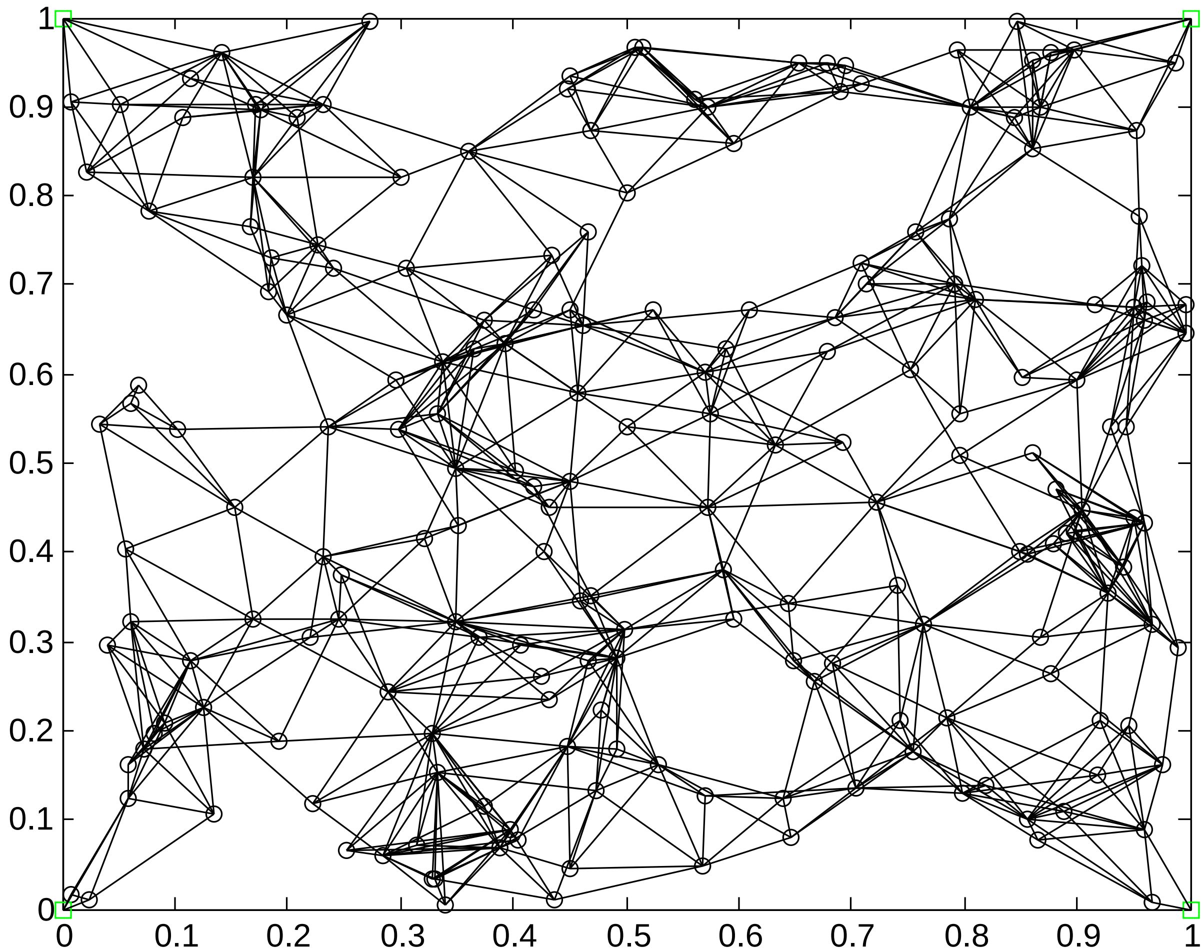
$m = 200$ sensors randomly distributed in $[0,1]^2$, $n - m = 4$ anchors at the corner of $[0,1]^2$, $\rho = 0.15$.



* : $\boldsymbol{c}^p$ = a computed location of censor p.

the true location $\boldsymbol{x}^p$ of sensor p is within $(\gamma^p)^{1/2} \leq 0.04$ from $\boldsymbol{c}^p$

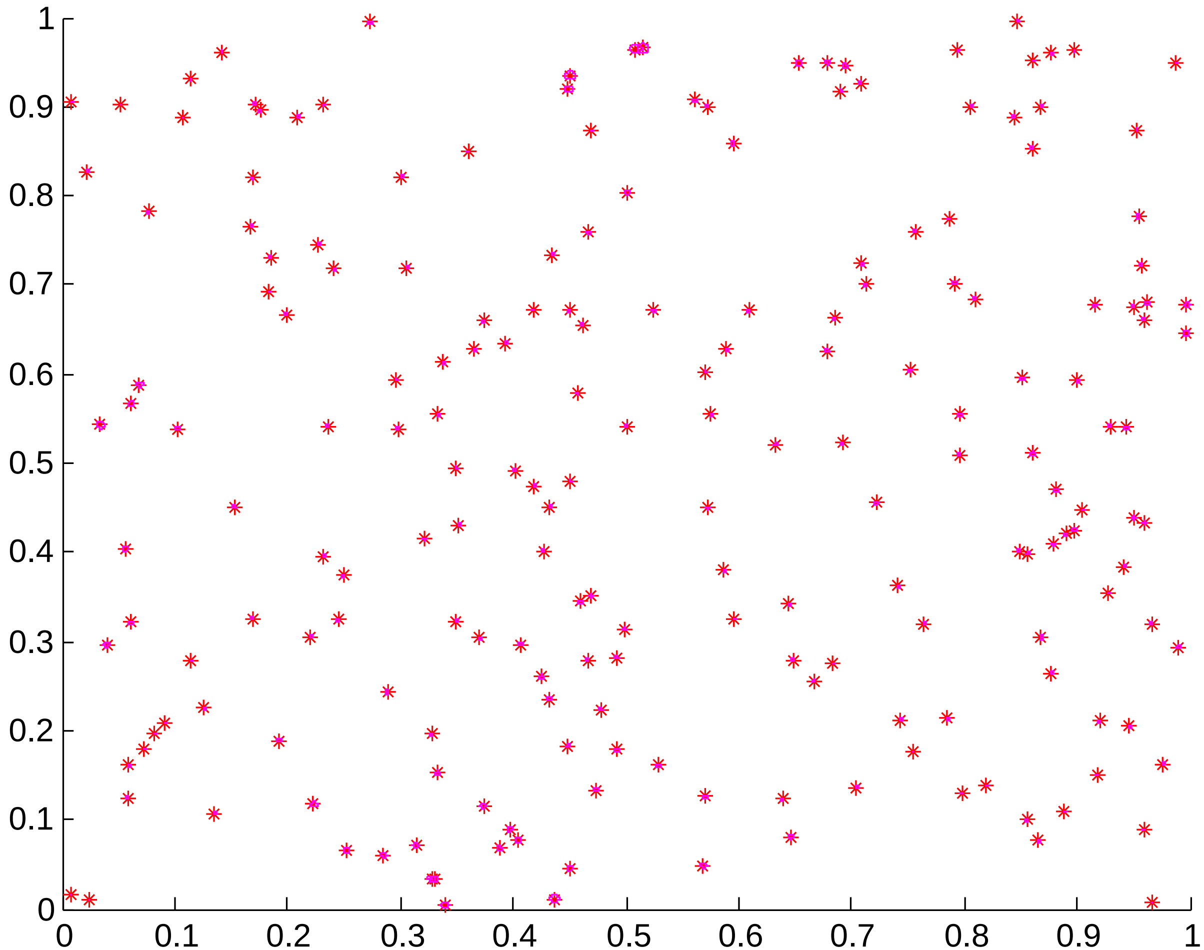
$m = 200$ sensors randomly distributed in $[0,1]^2$, $n - m = 4$ anchors at the corner of $[0,1]^2$, $\rho = 0.15$.



the true location $\boldsymbol{x}^p$ of sensor p

○——○ : the edge $(\boldsymbol{x}^p, \boldsymbol{x}^q)$ with a given exact distance

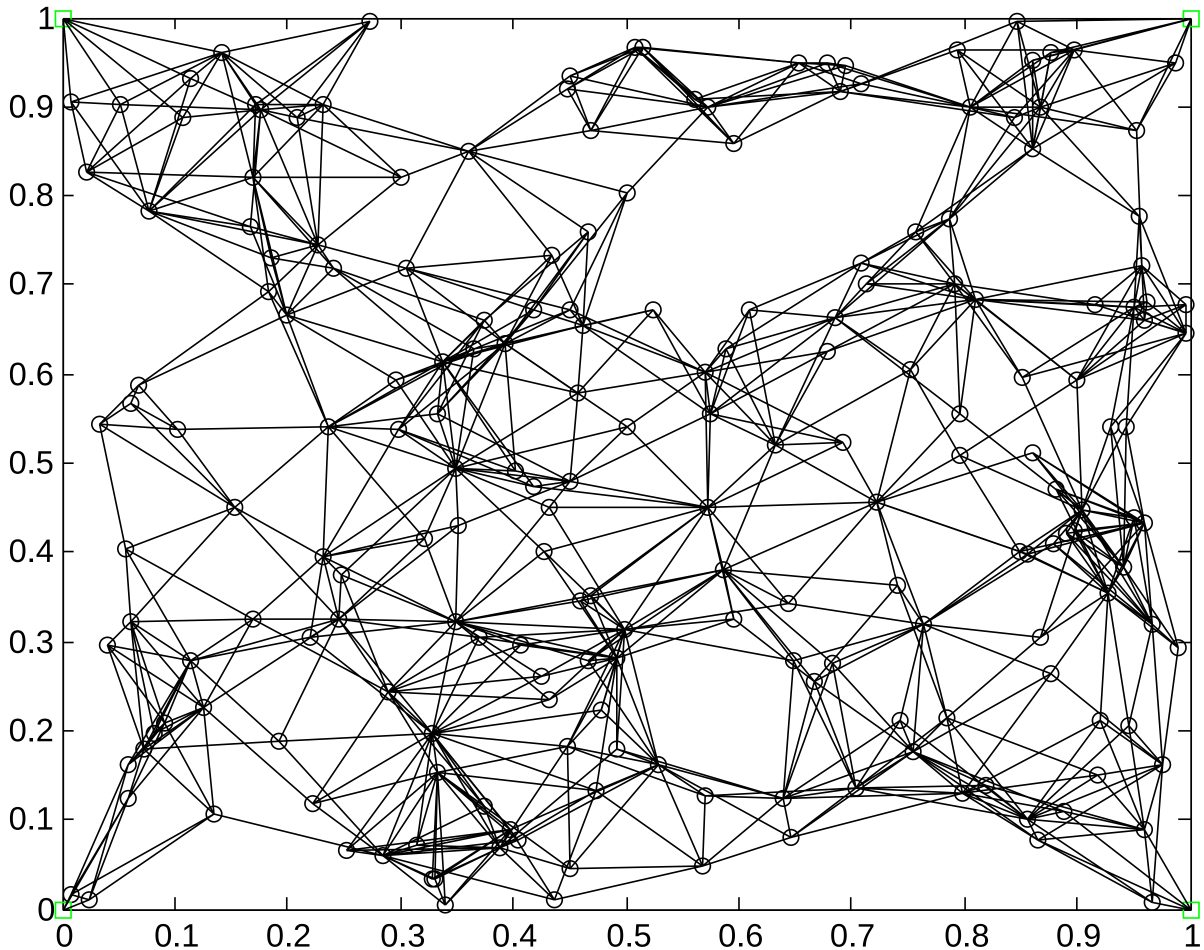
$m = 200$ sensors randomly distributed in $[0, 1]^2$, $n - m = 4$ anchors at the corner of $[0, 1]^2$, $\rho = 0.16$.



* : $\boldsymbol{c}^p$ = a computed location of censor p.

the true location $\boldsymbol{x}^p$ of sensor p is within $(\gamma^p)^{1/2} \leq 6.0$e-$3$ from $\boldsymbol{c}^p$

$m = 200$ sensors randomly distributed in $[0,1]^2$, $n - m = 4$ anchors at the corner of $[0,1]^2$, $\rho = 0.16$.



the true location $\boldsymbol{x}^p$ of sensor p

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Outline

1 Problem and Some Formulations

2 Theory: Lifting and SDP Relaxation

3 Numerical Results

4 Applications to the Sensor Network Localization Problem

5 Concluding Remarks

Concluding Remarks

- We can apply the proposed method to sensor network localization problems with inexact distance involving measurement error, but the results are not sharp.

- Polynomial optimization problems with a 0-1 variable x to determine whether $x = 0$ or $x = 1$.