Parallel Computing for Semidefinite Programming

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This presentation is based on a joint work "SDPA project" with

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1. SDP (semidefinite program) and its dual.

2. Existing numerical methods for SDPs.

3. A primal-dual IPM and outline of its parallel implementation.

4. Numerical results on SDPARA.

5. Positive definite matrix completion used in SDPARA-C.

6. Numerical results on SDPARA-C.

7. Comparison between SDPARA-C and SDPARA.

8. Concluding remarks

$$\mathcal{P}: \min \quad A_0 \bullet X \quad \text{sub.to} \quad A_p \bullet X = b_p \ (1 \leq p \leq m), \ \mathcal{S}^n \ni X \succeq O$$
$$\mathcal{D}: \max \quad \sum_{p=1}^m b_p y_p \quad \text{sub.to} \quad \sum_{p=1}^m A_p y_p + S = A_0, \ \mathcal{S}^n \ni S \succeq O$$

$$\begin{aligned}
\mathcal{S}^n &: \text{ the set of } n \times n \text{ symmetric matrices} \\
X, \ S \in \mathcal{S}^n, \ y_p \in R \ (1 \leq p \leq m) &: \text{ variables} \\
A_0, \ A_p \in \mathcal{S}^n, \ b_p \in R \ (1 \leq p \leq m) &: \text{ given data}
\end{aligned}$$

$$U \bullet V = \sum_{i=1}^n \sum_{j=1}^n U_{ij} V_{ij} \text{ for every } U, \ V \in R^{n \times n}$$

$$X \succeq O \Leftrightarrow X \in \mathcal{S}^n \text{ is positive semidefinite}$$
$$X \succ O \Leftrightarrow X \in \mathcal{S}^n \text{ is positive definite}$$

Important features — SDP can be large-scale easily

- $n \times n$ matrix variables $X, \ S \in \mathcal{S}^n$, each of which involves $n(n+1)/2$ real variables; for example, $n = 2000 \Rightarrow n(n+1)/2 \approx 2$ million.
- m linear equality constraints in $\mathcal{P}$ or m A_p's $\in \mathcal{S}^n$.

$\Downarrow$

◊ Exploit structured sparsity.
◊ Enormous computational power $\Rightarrow$ parallel computation.

$\mathcal{P}$: min $A_0 \bullet X$ sub.to $A_p \bullet X = b_p \ (1 \leq p \leq m)$, $\mathcal{S}^n \ni X \succeq O$
$\mathcal{D}$: max $\sum_{p=1}^m b_p y_p$ sub.to $\sum_{p=1}^m A_p y_p + S = A_0$, $\mathcal{S}^n \ni S \succeq O$

Structured sparsity

The aggregate sparsity pattern $\widehat{A}$: a symbolic $n \times n$ matrix:

$$\widehat{A}_{ij} = \begin{cases} \star & \text{if the } (i,j)\text{th element of } A_p \text{ is nonzero for } \exists p = 0,\ldots,m, \\ 0 & \text{otherwise,} \end{cases}$$

where $\star$ denotes a nonzero number.

Example: $m = 1$

$$A_0 = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ A_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix} \Rightarrow \widehat{A} = \begin{pmatrix} \star & \star & 0 & \star \\ \star & \star & \star & 0 \\ 0 & \star & \star & 0 \\ \star & 0 & 0 & \star \end{pmatrix}.$$

Next — three types of structured sparsity

$\mathcal{P}$: min $A_0 \bullet X$ sub.to $A_p \bullet X = b_p \ (1 \le p \le m)$, $\mathcal{S}^n \ni X \succeq O$
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where $\star$ denotes a nonzero number.

Structured sparsity-1 : $\widehat{A}$ is block-diagonal.

Then X, S has the same diagonal block structure as $\widehat{A}$.

$$\widehat{A} = \begin{pmatrix} B_1 & O & O \\ O & B_2 & O \\ O & O & B_3 \end{pmatrix}, \ B_i : \text{symmetric.}$$

Example: CH_3N : an SDP from quantum chemistry, Fukuda et al. 2005.

$m = 20,709$, $n = 12,802$, "the number of blocks in $\widehat{A}$" $= 22$,

the largest bl.size $= 3,211 \times 3,211$, the average bl.size $= 583 \times 583$.

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where $\star$ denotes a nonzero number.

Structured sparsity-2 : $\widehat{A}$ has a sparse Cholesky factorization.

"a small bandwidth" "a small bandwidth + bordered"

$$\widehat{A} = \begin{pmatrix} \star & \star & O & O & O \\ \star & \star & \star & O & O \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ O & O & \star & \star & \star \\ O & O & \cdots & \star & \star \end{pmatrix}, \quad \widehat{A} = \begin{pmatrix} \star & \star & O & O & \star \\ \star & \star & \star & O & \star \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ O & O & \star & \star & \star \\ \star & \star & \cdots & \star & \star \end{pmatrix}, \quad \star : \text{bl.matrix} \neq O$$

- S has the same sparsity pattern as $\widehat{A}$ but X is fully dense in general!

$\Rightarrow$ the positive definite matrix completion in SDPARA-C.

$\mathcal{P}$: min $A_0 \bullet X$ sub.to $A_p \bullet X = b_p \ (1 \leq p \leq m), \ \mathcal{S}^n \ni X \succeq O$

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Structured sparsity

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where $\star$ denotes a nonzero number.

Structured sparsity-3 : block-diagonal $\widehat{A}$ + blockwise orthogonality,

for most pairs (p, q) $1 \leq p < q \leq m$,

A_p and A_q do not share nonzero blocks; hence $A_p \bullet A_q = 0$.

$\Rightarrow$ the Schur complement matrix used in PDIPM becomes sparse.

$$A_1 = \begin{pmatrix} A_{11} & O & O \\ O & O & O \\ O & O & O \end{pmatrix}, \ A_2 = \begin{pmatrix} O & O & O \\ O & A_{22} & O \\ O & O & O \end{pmatrix}, \ A_3 = \begin{pmatrix} O & O & O \\ O & O & O \\ O & O & A_{33} \end{pmatrix}.$$

- An engineering application, Ben-Tal and Nemirovskii 1999.
- A sparse SDP relaxation of poly. opt. problem, Waki et al. 2005.
- Not incorporated in our software yet.

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2. **Existing numerical methods for SDPs.**
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Some existing numerical methods for SDPs

- **IPMs (Interior-point methods)**
  - Primal-dual scaling, CSDP(Borchers), SDPA(Fujisawa-K-Nakata), SDPT3(Todd-Toh-Tutuncu), SeDuMi(F.Sturm)
  - Dual scaling, DSDP(Benson-Ye)
- **Nonlinear programming approaches**
  - Spectral bundle method(Helmberg-Kiwiel)
  - Gradient-based log-barrier method(Burer-Monteiro-Zhang)
  - PENON(M. Kocvara) — Generalized augmented Lagrangian method
  - Saddle point mirror-prox algorithm (Lu-Nemirovski-Monteiro)

- Medium scale SDPs (e.g. $n, \ m = 1000$) and high accuracy.
- Large scale SDPs (e.g., n=10,000) and low accuracy.

- Parallel implementation of SDPA, DSDP, Spectral bundle method
- SDPARA, SDPARA-C to solve large-scale SDPs with high accuracy.

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Advantages of primal-dual IPMs (interior-point methods)

- **Highly accurate solutions.** *cf* **S.bundle and Gradient-based methods**
- **The number of iterations is small;**
  usually 20 — 100 iterations in practice, independent of sizes of SDPs.

Disadvantage of primal-dual IPMs (interior-point methods)

- **Heavy computation in each iteration**

$\Downarrow$

Parallel execution of heavy computation in each iteration

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Generic primal-dual IPM on a single CPU ⇒ SDPA

Step 0: Choose $(X, y, S) = (X^0, y^0, S^0)$; $X^0 \succ O$ and $S^0 \succ O$. $k = 1$.
Step 1: Compute a search direction (dX, dy, dS). ⇒ $B dy = r$
Step 2: Choose α_p and α_d;
$X^{k+1} = X^k + \alpha_p dX \succ O$, $S^{k+1} = S^k + \alpha_d dS \succ O$, $y^{k+1} = y^k + \alpha_d dy$.
Step 3: Let $k = k + 1$. Go to Step 1.

B : $m \times m$ fully dense except special cases, computed from A_p, X, S.

Major time consumption (second) on a single cpu implemention.

parallel comp.	part	control11	theta6	maxG51
SDPARA ⇐	Elements of B	463.2	78.3	1.5
SDPARA ⇐	Cholesky fact. of B	31.7	209.8	3.0
SDPARA-C ⇐	dX	1.8	1.8	47.3
SDPARA-C ⇐	Other dense mat. comp.	1.0	4.1	86.5
	Others	7.2	5.13	1.8
	Total	505.2	292.3	140.2

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SDPA $\Longrightarrow\Longrightarrow\Longrightarrow\Longrightarrow\Longrightarrow\Longrightarrow\Longrightarrow$ (parallel) SDPARA for large $m \leq 30,000$ but small $n \leq 2,000$

- Computation of Schur complement matrix B
- Cholesky factorization of B

$$\begin{array}{llll} \mathcal{P}: \min & A_0 \bullet X & \text{sub.to} & A_p \bullet X = b_p \ (1 \leq p \leq m), \ \mathcal{S}^n \ni X \succeq O \\ \mathcal{D}: \max & \sum_{p=1}^m b_p y_p & \text{sub.to} & \sum_{p=1}^m A_p y_p + S = A_0, \ \mathcal{S}^n \ni S \succeq O \end{array}$$

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SDPA $\Longrightarrow\Longrightarrow\Longrightarrow$ (parallel) $\Longrightarrow\Longrightarrow\Longrightarrow\Longrightarrow$ **SDPARA for large $m \leq 30,000$ but small $n \leq 2,000$**

$\Downarrow$

$\Downarrow$ • **positive definite matrix completion**

$\Downarrow$

$\Downarrow$

SDPARA-C for larger n

SDPARA + p.d. matrix completion = SDPARA-C.

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Numerical results on SDPARA

Hardware:

- PC cluster: 2×Opteron 2GHz cpus with 6GB memory in each node.
- MPI (Message Passing Interface) for communication between CPUs.
- Myrinet-2000 between nodes, 2 times faster than gigabit ethernet.

Software:

- ScaLAPACK for parallel Cholesky factorization.
- All data A_p, b_p are distributed to every node.
- Iterates $\{(X^k, y^k, S^k)\}$ are stored and updated in each node.
- Some heavy computations (B and its Cholesky factization to solve $Bdy = r$) are done in parallel and their results are distributed to all nodes, but all other computations are done individually and independently in each node.
- Primal and dual feasibilities, relative duality gaps $\le 1.0e^{-6}$.

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SDPs from quantum chemistry, Fukuda et al. 2005.

atoms/molecules	m	n	#blocks	the sizes of largest blocks
O	7230	5990	22	[1450, 1450, 450, . . .]
HF	15018	10146	22	[2520, 2520, 792, . . .]
CH_3N	20709	12802	22	[3211, 3211, 1014, . . .]

number of processors		16	64	128	256
O	elements of B	10100.3	2720.4	1205.9	694.2
	Chol.fact. of B	218.2	87.3	68.2	106.2
	total	14250.6	4453.3	3281.1	2951.6
HF	elements of B	*	*	13076.1	6833.0
	Chol.fact. of B	*	*	520.2	671.0
	total	*	*	26797.1	20780.7
CH_3N	elements of B	*	*	34188.9	18003.3
	Chol.fact. of B	*	*	1008.9	1309.9
	total	*	*	57034.8	45488.9

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The aggregate sparsity pattern matrix $\widehat{A}$: a symbolic $n \times n$ matrix:

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— the structured sparsity $\Longleftrightarrow$ a sparse Cholesky factorization

- $S = A_0 - \sum_{p=1}^m A_p y_p$ has the sparsity as $\widehat{A}$, but X does not.
- Using the positive matrix completion, we can make X^{-1} sparse!

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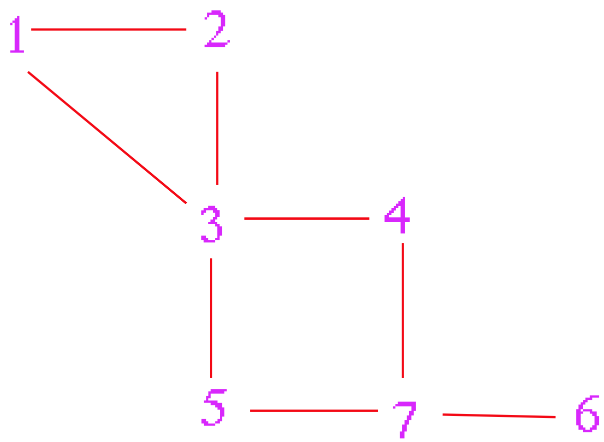
— **the structured sparsity $\Longleftrightarrow$ a sparse Cholesky factorization**

The aggregate sparsity pattern graph $G(N, E)$, where

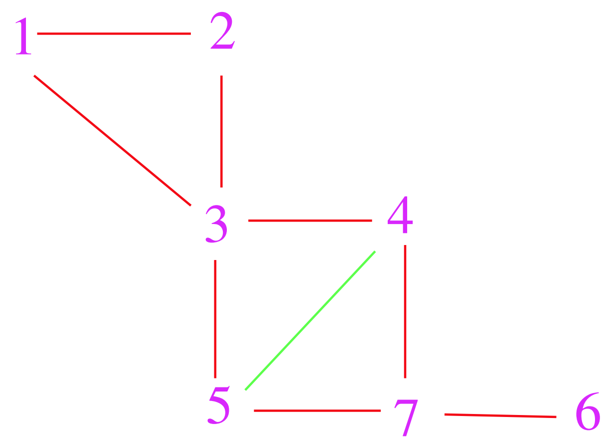
$$N = \{1, 2, \ldots, n\} \quad \text{and } E \equiv \{(i,j) : \widehat{A}_{ij} = \star\}$$

— **the structured sparsity $\Longleftrightarrow$ $\exists$ a sparse chordal extension**
($\forall$ minimal cycle has at most 3 edges)

$$\widehat{A} = \begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & \star & \star & \star & 0 & 0 & 0 & 0 \\ 2 & \star & \star & \star & 0 & 0 & 0 & 0 \\ 3 & \star & \star & \star & \star & \star & 0 & 0 \\ 4 & 0 & 0 & \star & \star & 0 & 0 & \star \\ 5 & 0 & 0 & \star & 0 & \star & 0 & \star \\ 6 & 0 & 0 & 0 & 0 & 0 & \star & \star \\ 7 & 0 & 0 & 0 & \star & \star & \star & \star \end{array}$$



$G(N, E)$: not chordal



$G(N, E')$: chordal

Example: $m = 2$, $n = 4$.

$$\min \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & 3 \\ 1 & 2 & 3 & 9 \end{pmatrix} \bullet \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{pmatrix}$$

Remember! $C \bullet X = \sum_{i,j} C_{ij} X_{ij}$

$$\text{sub.to} \begin{pmatrix} 9 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 7 & 3 \\ 1 & 2 & 3 & 5 \end{pmatrix} \bullet X = 6, \begin{pmatrix} 2 & 0 & 0 & 6 \\ 0 & 8 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 6 & 8 & 0 & 4 \end{pmatrix} \bullet X = 5, \ X \succeq O$$

- "the aggregate sparsity pattern" over all A_p's $E = \{(i, j)$ in Red $\}$
- X_{ij} $(i, j) \notin E$ are unnecessary to evaluate the objective function and the equality constraints, but necessary for $X \succeq O$.

pd matrix completion: Suppose $X_{ij} \in \mathbb{R}$ $((i, j) \in S)$ are given.

(a) $\exists X_{ij} \in \mathbb{R}$ $((i, j) \notin E)$; a completed matrix X is positive definite iff

$$\begin{pmatrix} X_{11} & X_{14} \\ X_{41} & X_{44} \end{pmatrix}, \begin{pmatrix} X_{11} & X_{24} \\ X_{42} & X_{44} \end{pmatrix}, \begin{pmatrix} X_{11} & X_{34} \\ X_{43} & X_{44} \end{pmatrix}$$ are positive definite.

(b) We can compute such a completed matrix X with the property that X^{-1} has the same sparsity pattern as E.

Example: $m = 2$, $n = 4$.

$$\min \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & 3 \\ 1 & 2 & 3 & 9 \end{pmatrix} \bullet \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{pmatrix}$$

Remember! $C \bullet X = \sum_{i,j} C_{ij} X_{ij}$

$$\text{sub.to} \begin{pmatrix} 9 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 7 & 3 \\ 1 & 2 & 3 & 5 \end{pmatrix} \bullet X = 6, \quad \begin{pmatrix} 2 & 0 & 0 & 6 \\ 0 & 8 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 6 & 8 & 0 & 4 \end{pmatrix} \bullet X = 5, \quad X \succeq O$$

- "the aggregate sparsity pattern" over all A_p's $E = \{(i, j)$ in Red $\}$
- X_{ij} $(i, j) \notin E$ are unnecessary to evaluate the objective function and the equality constraints, but necessary for $X \succeq O$.

- Using pd matrix completion, we can generate each iterate (X, y, S) such that both X^{-1} and S have the same sparsity pattern as E when E is "nicely sparse" as above; $G(N, E)$ forms a chordal graph.
- In general, we need to extend $G(N, E)$ to a chordal graph.

$$\mathcal{P}: \min \quad A_0 \bullet X \quad \text{sub.to} \quad A_p \bullet X = b_p \ (1 \leq p \leq m), \ \mathcal{S}^n \ni X \succeq O$$
$$\mathcal{D}: \max \quad \sum_{p=1}^m b_p y_p \quad \text{sub.to} \quad \sum_{p=1}^m A_p y_p + S = A_0, \ \mathcal{S}^n \ni S \succeq O$$

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Numerical results: large-size SDPs by SDPARA-C (64 CPUs)

Hardware:

- **PC cluster: Athlon 1900+ (1.6 GHz) cpu with 768 MB memory in each node**
- **MPI (Message Passing Interface) for communication between CPUs.**
- **Myrinet-2000 between nodes, 2 times faster than gigabit ethernet.**

Software:

- **SDPARA + the positive definite matrix completion.**
- **Primal and dual feasibilities, relative duality gaps $\leq 1.0e^{-7}$.**

$$\begin{array}{llll} \mathcal{P}: \min & A_0 \bullet X & \text{sub.to} & A_p \bullet X = b_p \ (1 \le p \le m), \ \mathcal{S}^n \ni X \succeq O \\ \mathcal{D}: \max & \sum_{p=1}^m b_p y_p & \text{sub.to} & \sum_{p=1}^m A_p y_p + S = A_0, \ \mathcal{S}^n \ni S \succeq O \end{array}$$

Large-size SDPs by SDPARA-C (64 CPUs)

3 types of test Problems:

(a) SDP relaxations of randomly generated max. cut problems on lattice graphs with size 10×1000, 10×2000 and 10×4000.

(b) SDP relaxations of randomly generated max. clique problems on lattice graphs with size 10×500, 10×1000 and 10×2000.

(c) Randomly generated norm minimization problems

$$\min. \left\| F_0 - \sum_{i=1}^{10} F_i y_i \right\| \ \text{sub.to} \ y_i \in \mathbb{R} \ (i = 1, 2, \ldots, 10)$$

where $F_i : 10 \times 9990, \ 10 \times 19990$ or 10×39990 and $\|G\|$ = the square root of the max. eigenvalue of $G^T G$.

In all cases, the aggregate sparsity pattern consists of one block and is very sparse.

$$\mathcal{P}: \min \quad A_0 \bullet X \quad \text{sub.to} \quad A_p \bullet X = b_p \ (1 \le p \le m), \ \mathcal{S}^n \ni X \succeq O$$
$$\mathcal{D}: \max \quad \sum_{p=1}^m b_p y_p \quad \text{sub.to} \quad \sum_{p=1}^m A_p y_p + S = A_0, \ \mathcal{S}^n \ni S \succeq O$$

Large-size SDPs by SDPARA-C (64 CPUs)

	Problem	n	m	time (s)	memory (MB)
	Cut(10×1000)	10000	10000	274.3	126
(a)	Cut(10×2000)	20000	20000	1328.2	276
	Cut(10×4000)	40000	40000	7462.0	720
	Clique(10×500)	5000	9491	639.5	119
(b)	Clique(10×1000)	10000	18991	3033.2	259
	Clique(10×2000)	20000	37991	15329.0	669
	Norm(10×9990)	10000	11	409.5	164
(c)	Norm(10×19990)	20000	11	1800.9	304
	Norm(10×39990)	40000	11	7706.0	583

$$\begin{array}{llll} \mathcal{P}: \min & A_0 \bullet X & \text{sub.to} & A_p \bullet X = b_p \ (1 \leq p \leq m), \ \mathcal{S}^n \ni X \succeq O \\ \mathcal{D}: \max & \sum_{p=1}^m b_p y_p & \text{sub.to} & \sum_{p=1}^m A_p y_p + S = A_0, \ \mathcal{S}^n \ni S \succeq O \end{array}$$

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Two types of sparse problems

- $m \leq n$, $\widehat{A}$: very sparse $\Longrightarrow$ SDPARA-C
- $m > n \Longrightarrow$ SDPARA

$$\mathcal{P}: \min \quad A_0 \bullet X \quad \text{sub.to} \quad A_p \bullet X = b_p \ (1 \leq p \leq m), \ \mathcal{S}^n \ni X \succeq O$$
$$\mathcal{D}: \max \quad \sum_{p=1}^m b_p y_p \ \text{sub.to} \quad \sum_{p=1}^m A_p y_p + S = A_0, \ \mathcal{S}^n \ni S \succeq O$$

Comparison between SDPARAC and SDPARA where $m \leq n$.

maxG51,sdplib, max cut
m=1000, n=1000

#cpu	SDPARA-C	SDPARA
1	545	175
4	195	176
16	75	174
64	62	176

qpG51, sdplib
m=1000, n=2000

#cpu	SDPARA-C	SDPARA
1	2034	M(970MB)
4	575	M
16	196	M
64	108	M

torusg3-15 dimacs max cut
m=3,375, n=3,375

SDPARA-C	SDPARA
10387	M(920MB)
3099	M
989	M
530	M

$$\mathcal{P}: \min \quad A_0 \bullet X \quad \text{sub.to} \quad A_p \bullet X = b_p \ (1 \le p \le m), \ \mathcal{S}^n \ni X \succeq O$$
$$\mathcal{D}: \max \quad \sum_{p=1}^m b_p y_p \ \text{sub.to} \quad \sum_{p=1}^m A_p y_p + S = A_0, \ \mathcal{S}^n \ni S \succeq O$$

Comparison between SDPARAC and SDPARA where $m > n$.

control10, m=1326, n=(100,50)

#cpu	SDPARA-C	SDPARA
1	27437	429
4	7488	128
16	2308	43
64	1036	22

theta6, m=4375, n=300

#cpu	SDPARA-C	SDPARA
1	2650	694
4	695	147
16	221	65
64	100	37

In pd matrix completion: Heavy overheads to compute

$$B_{ij} = A_i X A_j \bullet S^{-1} \ (1 \le i, j \le m).$$

- X and S^{-1} are not stored because they are dense.
- Sparse Cholesky factorizations of X^{-1} and S are stored and used to compute B_{ij}.

$$\mathcal{P}: \min \quad A_0 \bullet X \quad \text{sub.to} \quad A_p \bullet X = b_p \ (1 \leq p \leq m), \ \mathcal{S}^n \ni X \succeq O$$
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(a) Two types of parallel primal-dual interior-point methods for SDPs

- **Parallel implementation SDPARA of SDPA, suitable for large m and smaller n.**

	m	n (22 blocks)	16cpu	64cpu	128cpu	256cpu
O	7230	5990 (1450,...)	14251	4453	3281	2952
HF	15018	10146 (2520, ...)	*	*	26798	20781
CH_3N	20709	12802 (3211, ...)	*	*	57035	45489

- **SDPARA-C = SDPARA + pd matrix completion, suitable for larger n and $\widehat{A}$: very sparse.**

	m	n	1cpu	4cpu	16cpu	64cpu
cut(10×4000)	40,000	40,000				7,462
norm(10×39,990)	11	40,000				7,706
torusg3-15	3,375	3,375	10378	3099	989	530

$$\mathcal{P}: \min \quad A_0 \bullet X \quad \text{sub.to} \quad A_p \bullet X = b_p \ (1 \leq p \leq m), \ \mathcal{S}^n \ni X \succeq O$$
$$\mathcal{D}: \max \quad \sum_{p=1}^m b_p y_p \quad \text{sub.to} \quad \sum_{p=1}^m A_p y_p + S = A_0, \ \mathcal{S}^n \ni S \succeq O$$

(b) Future research

- **Parallel sparse Cholesky fact. of the Schur complement mat. B. This is necessary for sparse SDP relaxations of polynomial optimization problems.**
- **Distribution of the data matrices $A_0, \ldots, A_m$ among cpus to solve huge-scale SDPs.**