Exploiting sparsity in polynomial optimization problems

NONCONVEX PROGRAMMING: LOCAL and GLOBAL APPROACHES

Theory, Algorithms and Applications

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1. Polynomial Optimization Problems (POPs)
2. Semidefinite Programming (SDP) relaxations of POPs
3. How do we formulate structured sparsity?
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5. Exploiting free variables in primal-dual interior-point methods for LP, SDP and SOCP
6. Application to sensor network localization problems
7. Concluding remarks

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Notation and Symbols

$\mathbb{R}^n$: the n-dim Euclidean space.

$\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$: a vector variable.

$f_j(\boldsymbol{x})$: a multivariate polynomial in $\boldsymbol{x} \in \mathbb{R}^n$ $(j = 0, 1, \ldots, m)$.

$$\boxed{\text{POP: min } f_0(\boldsymbol{x}) \text{ sub.to } f_j(\boldsymbol{x}) \geq 0 \text{ or } = 0 \ (j = 1, \ldots, m).}$$

Example: $n = 3$

$$\begin{array}{rl}
\text{min} & f_0(\boldsymbol{x}) \equiv x_1^3 - 2x_1x_2^2 + x_1^2x_2x_3 - 4x_3^2 \\
\text{sub.to} & f_1(\boldsymbol{x}) \equiv -x_1^2 + 5x_2x_3 + 1 \geq 0, \\
 & f_2(\boldsymbol{x}) \equiv x_1^2 - 3x_1x_2x_3 + 2x_3 + 2 \geq 0, \\
 & f_3(\boldsymbol{x}) \equiv -x_1^2 - x_2^2 - x_3^2 + 1 \geq 0, \\
 & x_1(x_1 - 1) = 0 \text{ (0-1 integer)}, \\
 & x_2 \geq 0, \ x_3 \geq 0, \ x_2x_3 = 0 \text{ (complementarity)}.
\end{array}$$

Contents

$$\text{POP:} \quad \min \quad f_0(\boldsymbol{x}) \quad \text{sub.to} \quad f_j(\boldsymbol{x}) \geq 0 \ (j = 1, \ldots, m).$$

[1] Lasserre, “Global optimization with polynomials and the problems of moments”, *SIAM J. on Optim.* (2001).
[2] Parrilo, “Semidefinite programming relaxations for semialgebraic problems”, *Math. Prog.* (2003).

- primal approach $\Rightarrow$ a sequence of SDP relaxations.
- dual approach $\Rightarrow$ a sequence of SOS relaxations.

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- primal approach $\Rightarrow$ a sequence of SDP relaxations.
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Main features:

(a) Lower bounds for the optimal value.

(b) Convergence to global optimal solutions under assump.

(c) Each relaxed problem can be solved as an SDP; its size $\uparrow$ rapidly along "the sequence" as the size of POP $\uparrow$, the deg. of poly. $\uparrow$, and/or we require higher accuracy.

(d) Expensive to solve large scale POPs in practice.

$\Rightarrow$ Exploiting Sparsity.

POP: $\quad \min \quad f_0(\boldsymbol{x}) \quad$ sub.to $\quad f_j(\boldsymbol{x}) \geq 0 \ (j = 1, \ldots, m).$

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Exploiting sparsity to solve larger scale problem in practice

[3] Kobayashi-Kim-Kojima, “Correlative sparsity in primal-dual interior-point methods for LP, SDP and SOCP”, Sep. 2006
$\Rightarrow$ Section 3

[4] Waki-Kim-Kojima-Muramatsu, “SOS and SDP relaxations for POPs with Structured Sparsity", *SIAM J. on Optim* (2006)
$\Rightarrow$ Section 4

Exploiting equalities in dual (free variables in primal) SDPs

[5] Kobayashi-Nakata-Kojima, "A Conversion of an SDP Having free variables into the Standard Form SDP", *Comp. Optim. Appl.* (2007)
$\Rightarrow$ Section 5

$\Rightarrow$ Appl. to sensor network localization problems in Section 6

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POP: $\min \quad f_0(\boldsymbol{x}) \quad$ sub.to $\quad f_j(\boldsymbol{x}) \geq 0 \; (j = 1, \ldots, m).$

How do we exploit sparsity in POP?

$$\Downarrow$$

The answer depends on which methods we use to solve POP.

POP

$\Downarrow$ SDP relaxation (Lasserre 2001)

SDP $\Leftarrow$ Primal-Dual IPM (Interior-Point Method)

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$\Downarrow$ SDP relaxation (Lasserre 2001)

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We will assume a structured sparsity (correlative sparsity):

(a) A sparse SDP relaxation $\Rightarrow$ SDP of smaller size.

(b) SDP satisfies "a similar structured sparsity" under which Primal-Dual IPM works efficiently.

- Characterized in terms of a sparse Cholesky factorization
- Characterized in terms of a sparse chordal graph structure
- Useful to solve large-scale sparse POPs in practice

POP min. $f_0(\boldsymbol{x})$ s.t. $f_j(\boldsymbol{x}) \geq 0$ or $= 0$ $(j = 1, \ldots, m)$.

$\boldsymbol{H} f_0(\boldsymbol{x})$: the $n \times n$ Hessian mat. of $f_0(\boldsymbol{x})$,

$\boldsymbol{J}\boldsymbol{f}_*(\boldsymbol{x})$: the $m \times n$ Jacob. mat. of $\boldsymbol{f}_*(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \ldots, f_m(\boldsymbol{x}))^T$,

$\boldsymbol{R}$: the csp matrix, the $n \times n$ density pattern matrix of $\boldsymbol{I} + \boldsymbol{H} f_0(\boldsymbol{x}) + \boldsymbol{J}\boldsymbol{f}_*(\boldsymbol{x})^T \boldsymbol{J}\boldsymbol{f}_*(\boldsymbol{x})$ (no cancellation in '$+$').

$[\boldsymbol{J}\boldsymbol{f}_*(\boldsymbol{x})^T \boldsymbol{J}\boldsymbol{f}_*(\boldsymbol{x})]_{ij} \neq 0$ iff x_i and x_j are in a common constraint.

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$[\boldsymbol{J}\boldsymbol{f}_*(\boldsymbol{x})^T \boldsymbol{J}\boldsymbol{f}_*(\boldsymbol{x})]_{ij} \neq 0$ iff x_i and x_j are in a common constraint.

Example: $f_0(\boldsymbol{x}) = \sum_{k=1}^{6} (-x_k^2)$
$f_j(\boldsymbol{x}) = 1 - x_j^2 - 2x_{j+1}^2 - x_6^2$ $(j = 1, 2, \ldots, 5)$.

$$\text{the csp matrix } \boldsymbol{R} = \begin{pmatrix} \star & \star & 0 & 0 & 0 & \star \\ \star & \star & \star & 0 & 0 & \star \\ 0 & \star & \star & \star & 0 & \star \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \end{pmatrix}$$

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$[\boldsymbol{J} \boldsymbol{f}_*(\boldsymbol{x})^T \boldsymbol{J} \boldsymbol{f}_*(\boldsymbol{x})]_{ij} \neq 0$ iff x_i and x_j are in a common constraint.

POP : c-sparse (correlatively sparse) $\Leftrightarrow$ The $n \times n$ csp matrix $\boldsymbol{R} = (R_{ij})$ allows a symbolic sparse Cholesky factorization (under a row & col. ordering like a symmetric min. deg. ordering).

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tri-daig. + bordered

$\Downarrow$

No fill-in in Cholesky factorization

Contents

Sparse (SDP) relaxation = Lasserre (2001) + c-sparsity

POP min. $f_0(\boldsymbol{x})$ s.t. $f_j(\boldsymbol{x}) \geq 0$ or $= 0$ $(j = 1, \ldots, m)$, c-sparse.

$$\Downarrow$$

A sequence of c-sparse SDP relaxation problems depending on the relaxation order $r = 1, 2, \ldots$;

(a) Under a moderate assumption,
opt. sol. of SDP $\rightarrow$ opt sol. of POP as $r \rightarrow \infty$
(Lasserre 2006).

(b) $r = \lceil$"the max. deg. of poly. in POP"$/2\rceil + 0 \sim 3$ is usually large enough to attain opt sol. of POP in practice.

(c) Such an r is unknown in theory except $\exists$ special cases.

(d) The size of SDP increases as $r \rightarrow \infty$.

Example of Sparse SDP relaxation for POP with Inequalities

POP: min $\sum_{i=1}^{4}(-x_i^3)$ s.t. $-a_i \times x_i^2 - x_4^2 + 1 \geq 0 \; (i = 1, 2, 3)$.

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$\Updownarrow$ with the relaxation order $r = 2 \geq r_0 = \lceil 3/2 \rceil = 2$

poly.SDP:

min $\sum_{i=1}^{4}(-x_i^3)$

s.t. $(-a_i \times x_i^2 - x_4^2 + 1)(1, x_i, x_4)^T(1, x_i, x_4) \succeq \boldsymbol{O} \ \ i = 1, 2, 3,$

$(1, x_j, x_4, x_j^2, x_j x_4, x_4^2)^T(1, x_j, x_4, x_j^2, x_j x_4, x_4^2) \succeq \boldsymbol{O} \ j = 1, 2, 3.$

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Represent poly.SDP as

min $\sum_{\boldsymbol{\alpha} \in \mathcal{A}_0} g_0(\boldsymbol{\alpha})\boldsymbol{x}^{\boldsymbol{\alpha}}$ s.t. $\sum_{\boldsymbol{\alpha} \in \mathcal{A}_j} \boldsymbol{G}_j(\boldsymbol{\alpha})\boldsymbol{x}^{\boldsymbol{\alpha}} \succeq \boldsymbol{O}$ $j = 1, \ldots, 6,$

where $\mathcal{A}_j \subset \mathbb{Z}_+^4$ and $\boldsymbol{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4}$; $\boldsymbol{x}^{(1,2,1,0)} = x_1 x_2^2 x_3$.

Example of Sparse SDP relaxation for POP with Inequalities

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$\Downarrow$ Linearize by replacing each $\boldsymbol{x}^{\boldsymbol{\alpha}}$ by an indep. var. $y_{\boldsymbol{\alpha}}$; $\boldsymbol{x}^0$ by 1

SDP min $\displaystyle\sum_{\boldsymbol{\alpha} \in \mathcal{A}_0} g_0(\boldsymbol{\alpha}) y_{\boldsymbol{\alpha}}$ s.t. $\displaystyle\sum_{\boldsymbol{\alpha} \in \mathcal{A}_j} \boldsymbol{G}_j(\boldsymbol{\alpha}) y_{\boldsymbol{\alpha}} \succeq \boldsymbol{O} \ j = 1, \ldots, 6,$

which forms an SDP relaxation of POP.

Example of Sparse SDP relaxation for POPs with Equalities

POP: min $\sum_{i=1}^{4}(-x_i^3)$ s.t. $-a_i \times x_i^2 - x_4^2 + 1 = 0 \; i = 1, 2, 3$.

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POP: min $\sum_{i=1}^{4}(-x_i^3)$ s.t. $-a_i \times x_i^2 - x_4^2 + 1 = 0 \; i = 1,2,3.$

$\Updownarrow$ with the relaxation order $r = 2 \geq r_0 = \lceil 3/2 \rceil = 2$

poly.SDP:

min $\sum_{i=1}^{4}(-x_i^3)$

s.t. $(-a_i \times x_i^2 - x_4^2 + 1)(1, x_i, x_4, x_i^2, x_i x_4, x_4^2)^T = \mathbf{0} \;\; i = 1,2,3,$

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min $\sum_{\boldsymbol{\alpha}\in\mathcal{A}_0} g_0(\boldsymbol{\alpha})\boldsymbol{x}^{\boldsymbol{\alpha}}$ s.t. $\sum_{\boldsymbol{\alpha}\in\mathcal{A}_i} \boldsymbol{g}_i \boldsymbol{x}^{\boldsymbol{\alpha}} = \mathbf{0}$

$\sum_{\boldsymbol{\alpha}\in\mathcal{A}_j} \boldsymbol{G}_j(\boldsymbol{\alpha})\boldsymbol{x}^{\boldsymbol{\alpha}} \succeq \boldsymbol{O} \; j = 1,2,3$

$\Downarrow$ Linearize by replacing each $\boldsymbol{x}^{\boldsymbol{\alpha}}$ by an indep. var. $y_{\boldsymbol{\alpha}}$; $\boldsymbol{x}^0$ by 1

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$\sum_{\boldsymbol{\alpha} \in \mathcal{A}_j} \boldsymbol{G}_j(\boldsymbol{\alpha}) \boldsymbol{y}_{\boldsymbol{\alpha}} \succeq \boldsymbol{O} \; j = 1, 2, 3$

Equalities in dual SDP $\Leftrightarrow$ Free variables in primal SDP $\Rightarrow$ next

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- How to handle free variables is an important issue in primal-dual interior-point methods for SDPs.
- Some methods have been developed;
  free $z = z_+ - z_-$, $z_+, \ z_- \geq 0$, using a second order cone.

A new method $\Rightarrow$

Primal SDP having free vector variable $\boldsymbol{z}$

$$\mathcal{P}: \quad \begin{array}{ll} \text{min} & \boldsymbol{d}^T\boldsymbol{z} + \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{D}\boldsymbol{z} + \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \succeq \mathbf{0}, \end{array} \qquad \begin{array}{l} \boldsymbol{D} : m \times k,\ \text{rank}\ \boldsymbol{D} = k, \\ \boldsymbol{A} : m \times n,\ \text{where}\ m \geq k \end{array}$$

Dual SDP having equality constraints

$$\mathcal{D}: \quad \text{max} \quad \boldsymbol{b}^T\boldsymbol{y} \quad \text{s.t.} \quad \boldsymbol{D}^T\boldsymbol{y} = \boldsymbol{d},\ \boldsymbol{A}^T\boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c},\ \boldsymbol{s} \succeq \mathbf{0}.$$

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- Primal approach: Eliminate free variable $\boldsymbol{z}$ by pivoting $\Rightarrow$

$$\widehat{\mathcal{P}}: \quad \begin{array}{ll} \min & \hat{\boldsymbol{c}}^T\boldsymbol{x} + \hat{\gamma} \\ \text{s.t.} & \widehat{\boldsymbol{A}}_2\boldsymbol{x} = \hat{\boldsymbol{b}}_2,\ \boldsymbol{x} \succeq \boldsymbol{0}, \end{array} \qquad \widehat{\boldsymbol{A}}_2 : (m-k) \times n.$$

- Dual : Solve $\boldsymbol{D}^T\boldsymbol{y} = \boldsymbol{d}$ in $\boldsymbol{y}_1$, $\boldsymbol{y} = (\boldsymbol{y}_1, \boldsymbol{y}_2) \in \mathbb{R}^{k+(m-k)}$.

$$\widehat{\mathcal{D}}: \quad \max \quad \hat{\boldsymbol{b}}_2^T\boldsymbol{y}_2 + \hat{\gamma} \quad \text{s.t.} \quad \widehat{\boldsymbol{A}}_2^T\boldsymbol{y}_2 + \boldsymbol{s} = \hat{\boldsymbol{c}},\ \boldsymbol{s} \succeq \boldsymbol{0}.$$

- The size gets smaller, but $\widehat{\boldsymbol{A}}_2$ could get denser than $\boldsymbol{A}$.
- Numerical stability in pivoting or solving $\boldsymbol{D}^T\boldsymbol{y} = \boldsymbol{d}$ in $\boldsymbol{y}_1$.

Primal SDP having free vector variable $\boldsymbol{z}$

$$\mathcal{P}: \quad \begin{array}{ll} \text{min} & \boldsymbol{d}^T\boldsymbol{z} + \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{D}\boldsymbol{z} + \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \succeq \mathbf{0}, \end{array} \qquad \begin{array}{l} \boldsymbol{D} : m \times k,\ \text{rank}\ \boldsymbol{D} = k, \\ \boldsymbol{A} : m \times n,\ \text{where}\ m \geq k \end{array}$$

Primal SDP having free vector variable $\boldsymbol{z}$

$$\mathcal{P}: \quad \begin{array}{ll} \text{min} & \boldsymbol{d}^T\boldsymbol{z} + \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{D}\boldsymbol{z} + \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \succeq \boldsymbol{0}, \end{array} \qquad \begin{array}{l} \boldsymbol{D} : m \times k, \ \text{rank } \boldsymbol{D} = k, \\ \boldsymbol{A} : m \times n, \ \text{where } m \geq k \end{array}$$

A stable sparse LU factorization to $\boldsymbol{D}$ for simplisity,

$$\boldsymbol{P}\boldsymbol{D}\boldsymbol{Q} \ = \ \boldsymbol{L}\boldsymbol{U} \quad \text{or} \quad \boldsymbol{D} = \boldsymbol{P}^T\boldsymbol{L}\boldsymbol{U}\boldsymbol{Q}^T \qquad = \boldsymbol{L}\boldsymbol{U}$$

$$k, \qquad \boldsymbol{U} \ : \ k \times k \text{ upper triangular},$$

$$\boldsymbol{L} \ = \ \begin{array}{cc} \begin{pmatrix} \boldsymbol{L}_1 \\ \boldsymbol{L}_2 \end{pmatrix} & \begin{array}{c} k \\ m-k \end{array} \end{array}, \ \boldsymbol{L}_1 \ : \ \text{lower triangular},$$

$$\boldsymbol{P} \ : \ \text{an } m \times m \text{ permutation matrix}, \qquad = \boldsymbol{I}$$

$$\boldsymbol{Q} \ : \ \text{a } k \times k \text{ permutation matrix}, \qquad = \boldsymbol{I}$$

Primal SDP having free vector variable $\boldsymbol{z}$

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A stable sparse LU factorization to $\boldsymbol{D}$

$$\boldsymbol{D} = \boldsymbol{L}\boldsymbol{U}$$

$$\begin{array}{ll} k, & \boldsymbol{U} : k \times k \text{ upper triangular,} \end{array}$$

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$$\widehat{\mathcal{P}}: \quad \begin{array}{ll} \text{min} & \hat{\boldsymbol{c}}^T\boldsymbol{x} + \hat{\gamma} \\ \text{s.t.} & \widehat{\boldsymbol{A}}_2\boldsymbol{x} = \hat{\boldsymbol{b}}_2,\ \boldsymbol{x} \succeq \boldsymbol{0},\ \boldsymbol{z} = \boldsymbol{U}^{-1}(\hat{\boldsymbol{b}}_1 - \widehat{\boldsymbol{A}}_1\boldsymbol{x}). \end{array} \qquad \widehat{\boldsymbol{A}}_2 : (m-k) \times n,\ \widehat{\boldsymbol{A}}_1 : k \times n$$

$$\hat{\boldsymbol{c}} = \boldsymbol{c} - \widehat{\boldsymbol{A}}_1^T\boldsymbol{U}^{-T}\boldsymbol{d},\ \hat{\gamma} = \hat{\boldsymbol{b}}_1^T\boldsymbol{U}^{-T}\boldsymbol{d},$$

$$\begin{pmatrix} \widehat{\boldsymbol{A}}_1 \\ \widehat{\boldsymbol{A}}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{L}_1 & \boldsymbol{O} \\ \boldsymbol{L}_2 & \boldsymbol{I} \end{pmatrix}^{-1} \boldsymbol{A},\ \begin{pmatrix} \hat{\boldsymbol{b}}_1 \\ \hat{\boldsymbol{b}}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{L}_1 & \boldsymbol{O} \\ \boldsymbol{L}_2 & \boldsymbol{I} \end{pmatrix}^{-1} \boldsymbol{b},$$

Primal SDP having free vector variable $\boldsymbol{z}$

$$\mathcal{P}: \quad \begin{array}{ll} \text{min} & \boldsymbol{d}^T\boldsymbol{z} + \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{D}\boldsymbol{z} + \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \succeq \boldsymbol{0}, \end{array} \qquad \begin{array}{l} \boldsymbol{D} : m \times k, \ \text{rank } \boldsymbol{D} = k, \\ \boldsymbol{A} : m \times n, \ \text{where } m \geq k \end{array}$$

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- k is larger $\Rightarrow$ smaller size
- $\boldsymbol{LU}$ factorization is well-conditioned $\Rightarrow$ higher accuracy
- $\boldsymbol{LU}$ factorization (or $\widehat{\boldsymbol{A}}_2$) is sparser $\Rightarrow$ more efficient
- can be applied to LP and SOCP

Contents

- All the methods described in Sections 3, 4 and 5 are applied in this section.
- Ongoing joint work with Kim and Waki

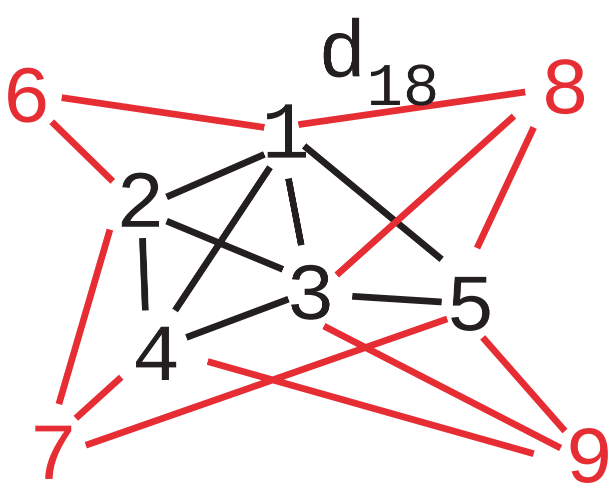
Sensor network localization problem: Let $s = 2$ or 3.

$$\begin{aligned} \boldsymbol{x}^p \in \mathbb{R}^s \quad &: \quad \text{unknown location of sensors } (p = 1, 2, \ldots, m), \\ \boldsymbol{x}^r = \boldsymbol{a}^r \in \mathbb{R}^s \quad &: \quad \text{known location of anchors } (r = m+1, \ldots, n), \\ d_{pq} \quad &= \quad \|\boldsymbol{x}^p - \boldsymbol{x}^q\| + \epsilon_{pq} \text{ — given for } (p, q) \in \mathcal{N}, \\ \mathcal{N} \quad &= \quad \{(p, q) : \|\boldsymbol{x}^p - \boldsymbol{x}^q\| \le \rho = \text{a given radio range}\} \end{aligned}$$

Here ϵ_{pq} denotes a noise.

$m = 5, \ n = 9$.
$1, \ldots, 5$: sensors
$6, 7, 8, 9$: anchors

Anchors' positions are fixed.
A distance is given for $\forall$ edge.
Compute locations of sensors.



- SDP relaxations Biswas et al. ’06, Nie ’06, ... for $s = 2$.
- An SOCP relaxation Tseng ’07 for $s = 2$.
- .....

$\Rightarrow$ Exploiting correlative sparsity in our new SDP relaxation

Basic idea of Sparse SDP relaxation

QOP: min $\sum_{pq}(v_{pq} - d_{pq})^2 \quad \equiv 0$

s.t $v_{pq}^2 = \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 \ (p,q) \in \mathcal{N}, \ \boldsymbol{x}^r = \boldsymbol{a}^r \ (r > m),$

$0 \leq (1-\gamma)d_{pq} \leq v_{pq} \leq (1+\delta)d_{pq} \ (p,q) \in \mathcal{N}.$

Here $0 \leq \gamma \leq 1$, $0 \leq \delta$; $\quad \gamma = \delta = 0$ or $d_{pq} = v_{pq}$ if $\epsilon_{pq} \equiv 0$

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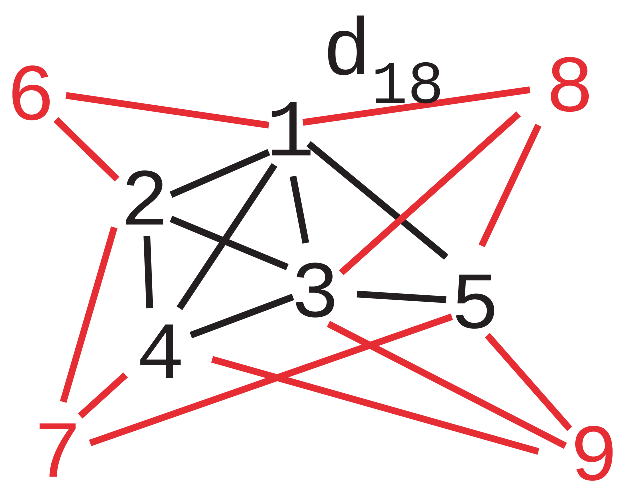

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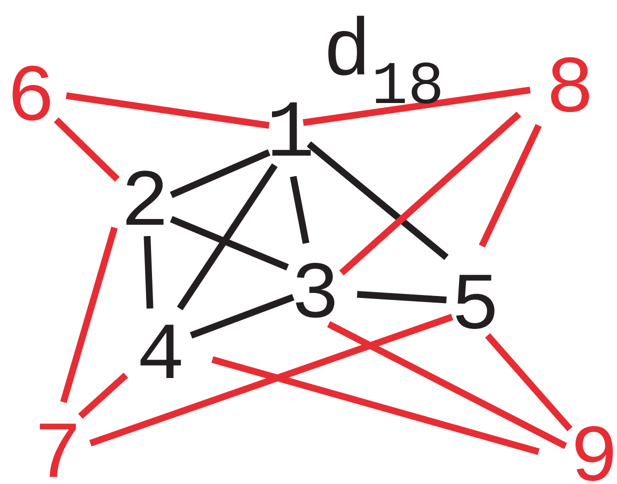

Anchors' positions are fixed.
A distance is given for $\forall$ edge.
Compute locations of sensors.

- Remove some edges to reduce the size.
- Keep red edges in this example.
- Remove black edges as long as deg. of $\forall$ node $\geq \delta$; $\delta = 4$ or 5.

Basic idea of $\boxed{\text{Sparse SDP relaxation}}$

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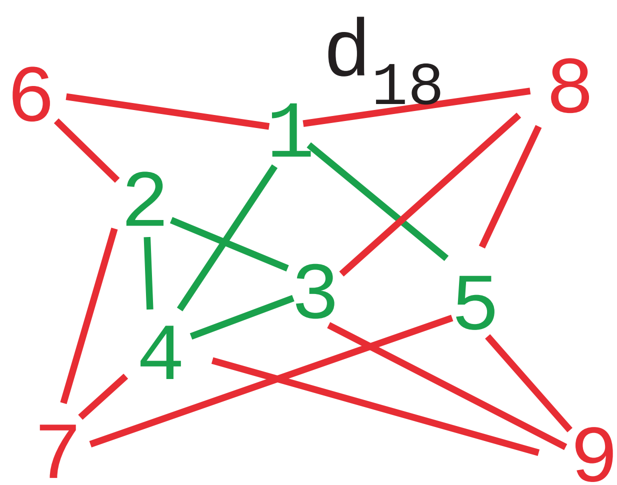

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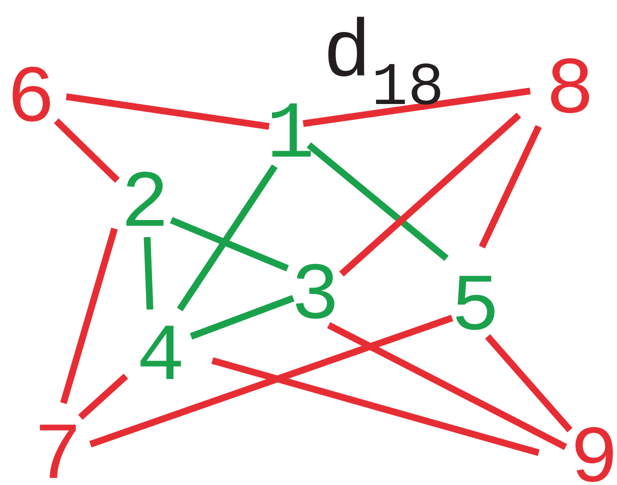


- Remove some edges to reduce the size.
- Keep red edges in this example.
- Remove black edges as long as deg. of $\forall$ node $\geq \delta$; $\delta = 4$ or 5.
- Use the red & green edges for $\mathcal{N}$ $\Rightarrow$ c-sparsity in QOP
- How we select the red & green edges for $\mathcal{N}$ is essential.

Preliminary numerical results

- Software — SparsePOP† + SeDuMi + Nonlinear LS meth.
- CPU 2.66 GHz Dual-Core Intel Xeon, memory 4 GB

† : Waki, S. Kim, M. Kojima and M. Muramatsu
"SparsePOP : a Sparse Semidefinite Programming Relaxation of Polynomial Optimization Problems"
March 2005. Revised August 2007.

Nonlinear LS method

to refine solutions computed by SparsePOP

$\Uparrow$

MATLAB function lsqnonlin

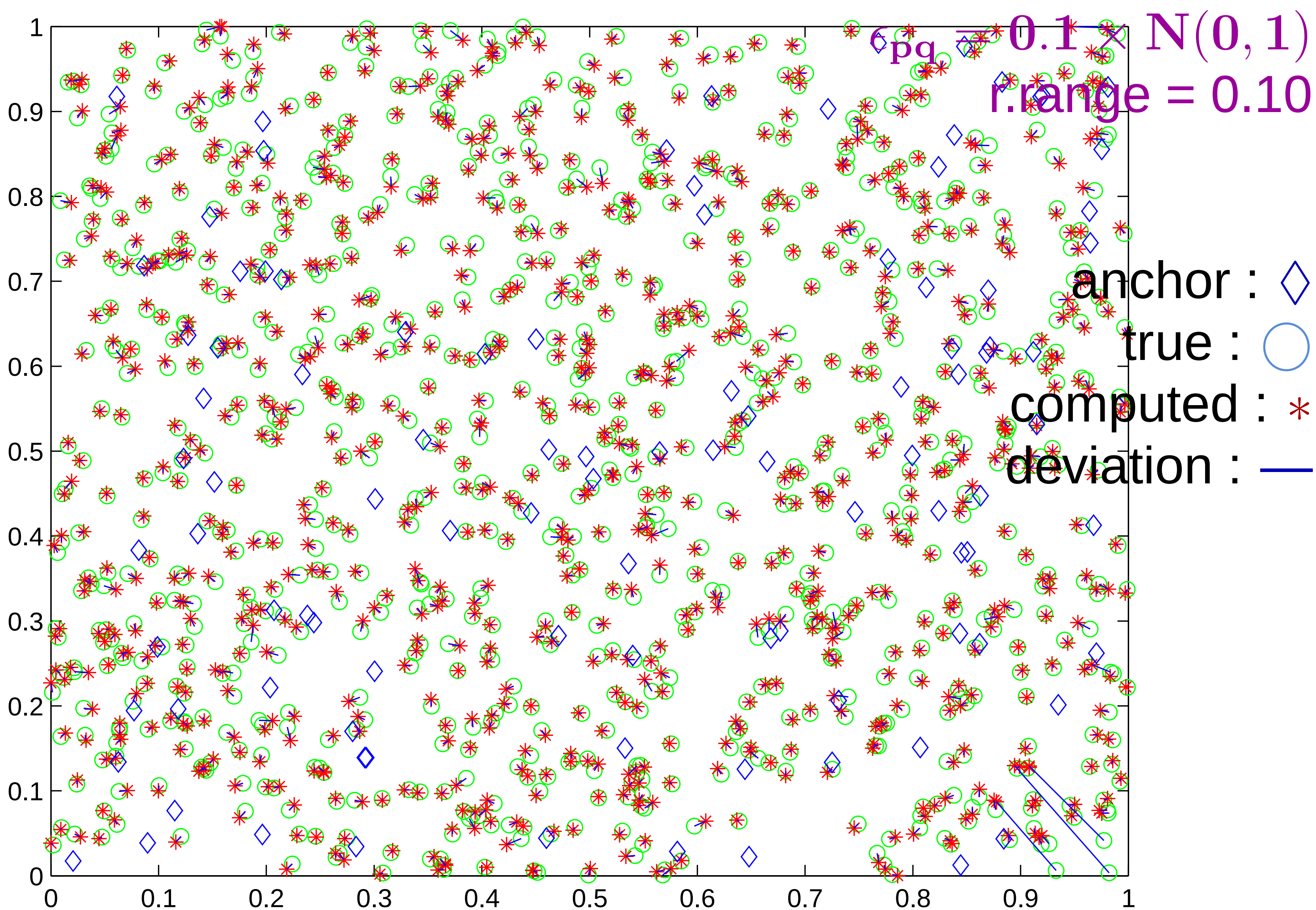
900 sensors and 100 anchors randomly distributed on $[0,1] \times [0,1]$

radio	No Noise: $\epsilon_{pq} = 0$		Noisy: $\epsilon_{pq} \sim 0.1 \times N(0,1)$	
range	rmsd	cpu	rmsd	cpu
0.1	3.1e-09	25.4	3.3e-04	204.9
0.2	1.1e-09	8.5	4.5e-04	173.8

rmsd $= \frac{1}{m}\left(\sum_{p=1}^{m} \|\boldsymbol{x}^p - \boldsymbol{a}^p\|^2\right)^{1/2}$ (root mean square distance)

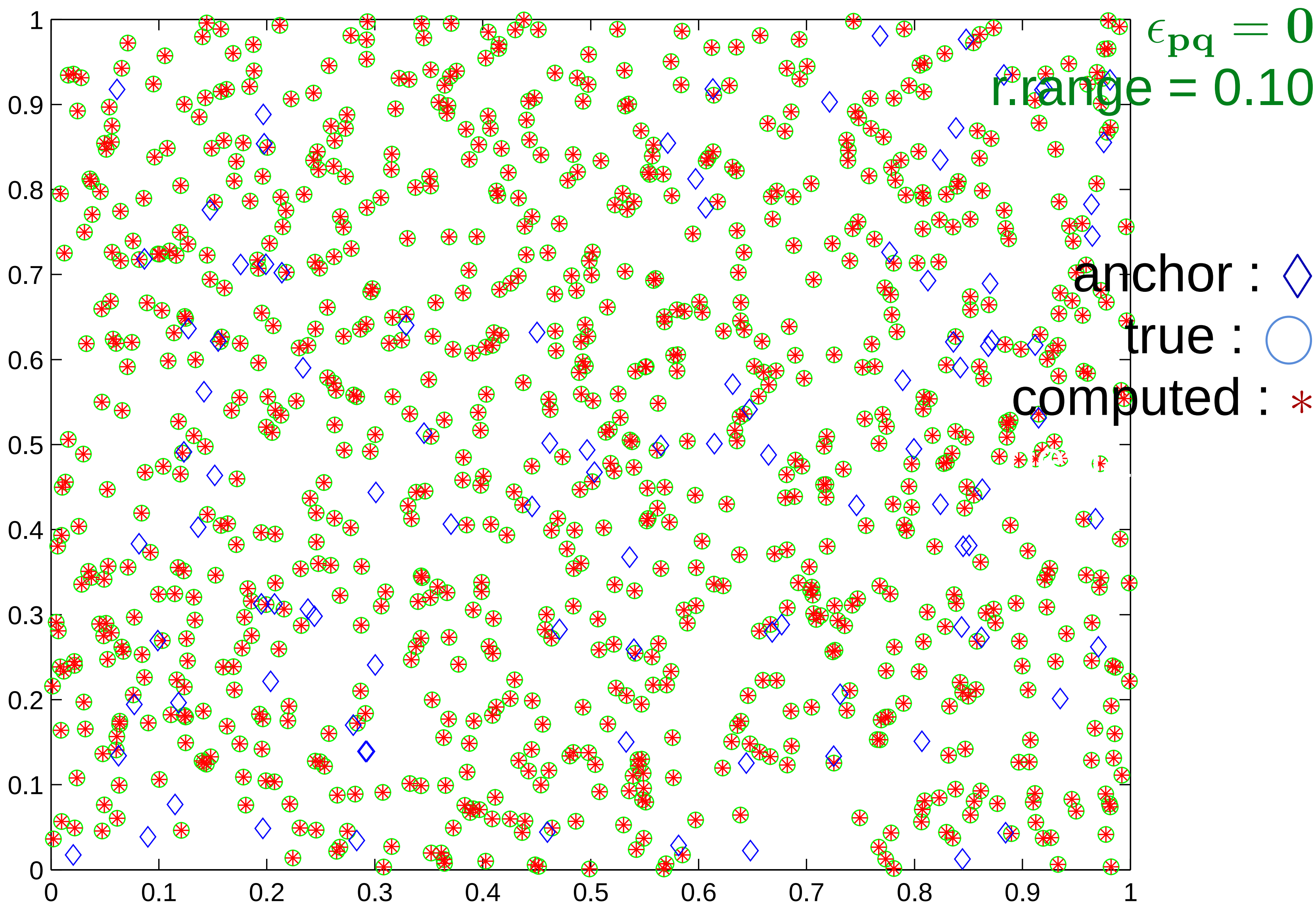
cpu = SeDuMi cpu time in second ($\not\supset$ conversion time)

900 sensors and 100 anchors on $[0,1] \times [0,1]$



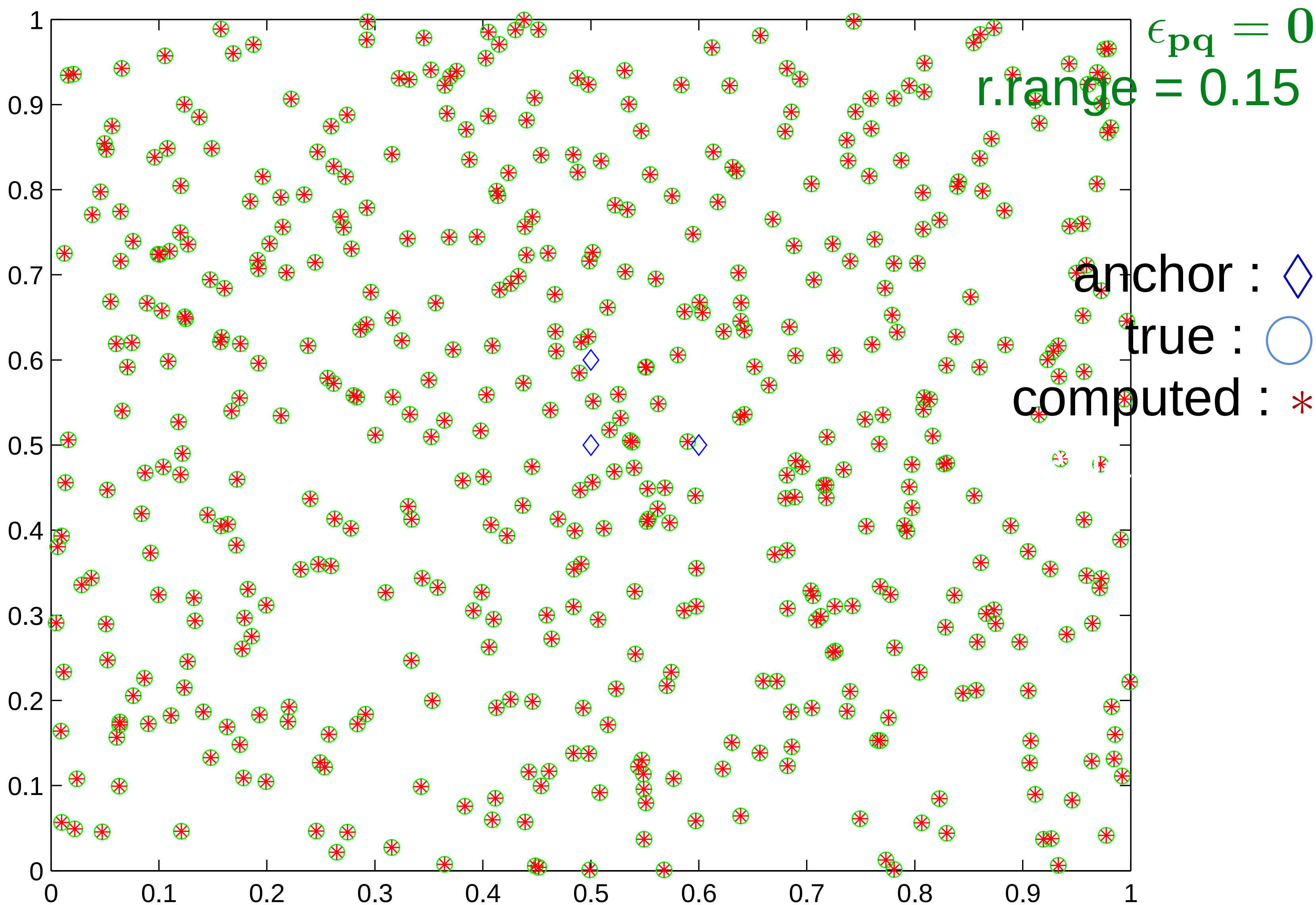
204.9 sec., rmsd $= \frac{1}{m}\left(\sum_{p=1}^{m} \|\boldsymbol{x}^p - \boldsymbol{a}^p\|^2\right)^{1/2}$ = 3.3e-04

900 sensors and 100 anchors on $[0,1] \times [0,1]$



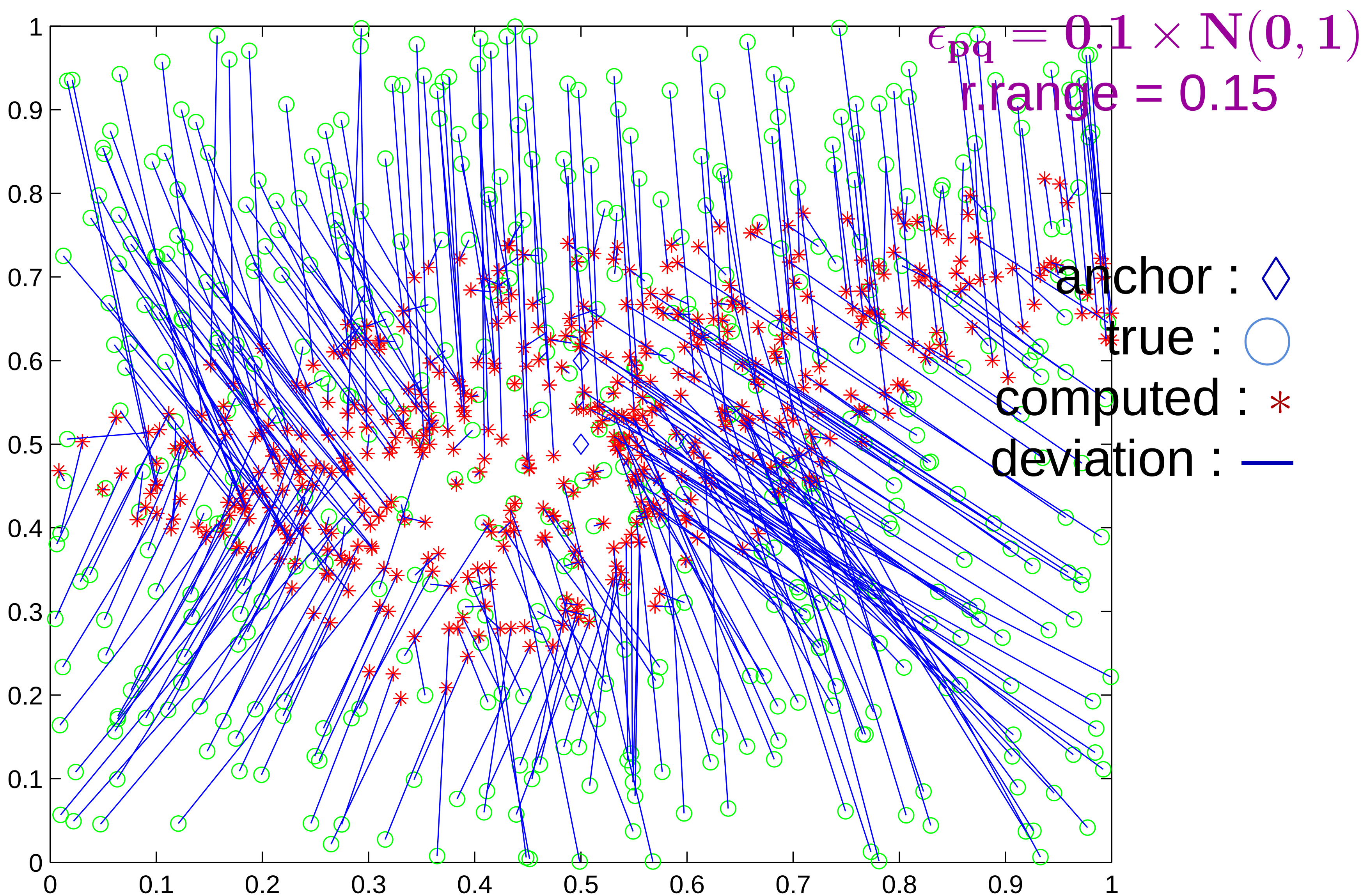
25.4 sec., rmsd $= \frac{1}{m}\left(\sum_{p=1}^{m} \|\boldsymbol{x}^p - \boldsymbol{a}^p\|^2\right)^{1/2}$ = 3.1e-09

500 sensors and 3 anchors at $(\mathbf{0.5, 0.5})$, $(\mathbf{0.6, 0.5})$, $(\mathbf{0.5, 0.6})$



320.0 sec., rmsd $= \frac{1}{m}\left(\sum_{p=1}^{m}\|\boldsymbol{x}^p - \boldsymbol{a}^p\|^2\right)^{1/2}$ = 5.7e-09

500 sensors and 3 anchors at $(0.5, 0.5),\ (0.6, 0.5),\ (0.5, 0.6)$



1324.5 sec., rmsd $= \dfrac{1}{m}\left(\sum_{p=1}^{m} \|\boldsymbol{x}^p - \boldsymbol{a}^p\|^2\right)^{1/2}$ = 1.2e-02

100 sensors and 27 anchors on $3 \times 3 \times 3$ grid in $[0,1]^3$

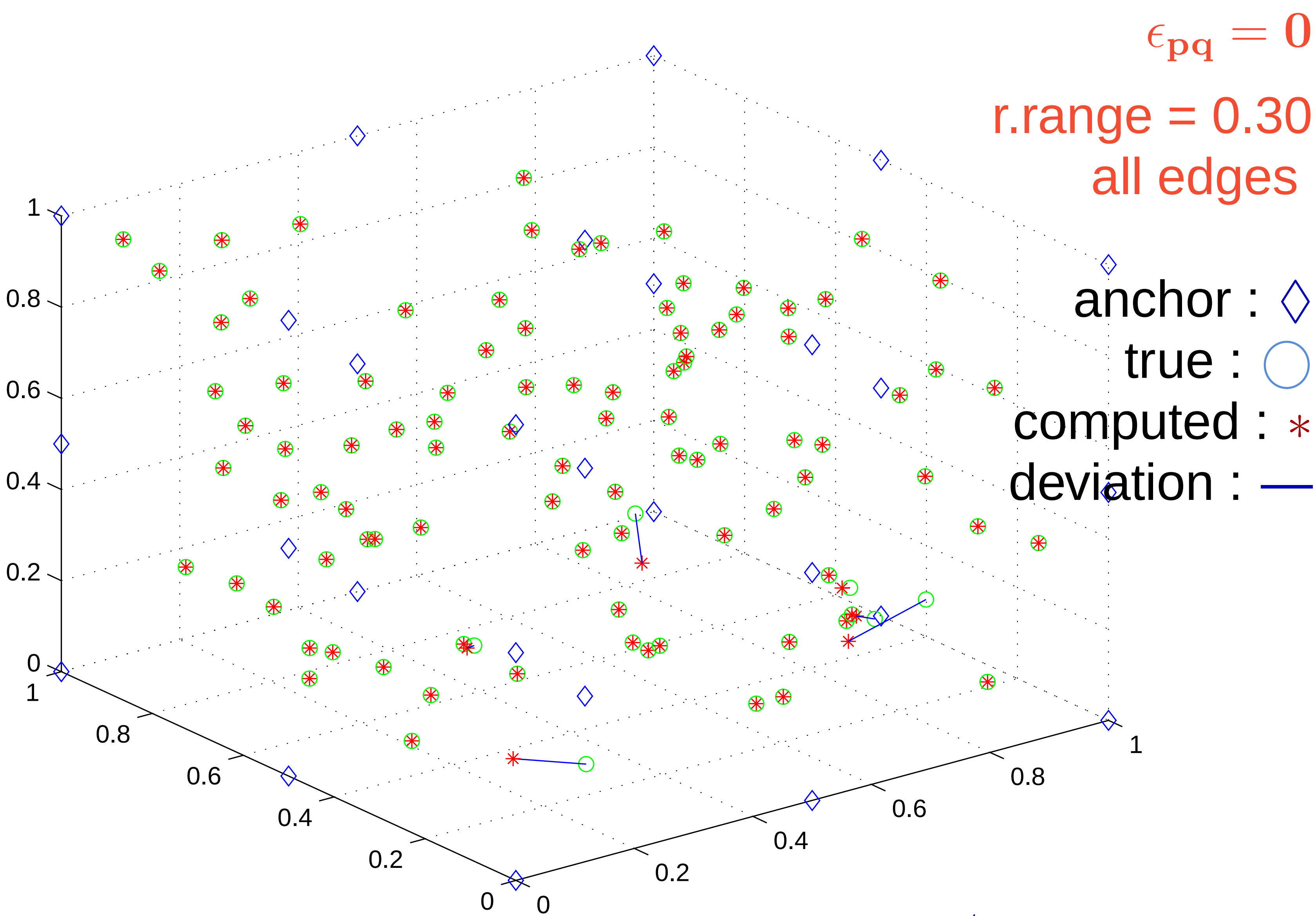
	$\epsilon_{pq} = 0$		$\epsilon_{pq} \sim 0.1 \times N(0,1)$	
radio range	rmsd	cpu	rmsd	cpu
0.25	2.8e-02	5.6	2.0e-02	10.3
0.30	3.4e-03	16.8	8.2e-03	19.8
0.30, all edges	3.4e-03	267.9		
0.35	3.7e-09	9.9	4.5e-03	14.4
0.40	2.2e-09	4.6	4.4e-03	11.8

cpu = SeDuMi cpu time in second ($\not\supset$ conversion time)

rmsd $= \frac{1}{m}\left(\sum_{p=1}^{m} \|\boldsymbol{x}^p - \boldsymbol{a}^p\|^2\right)^{1/2}$ (root mean square distance)

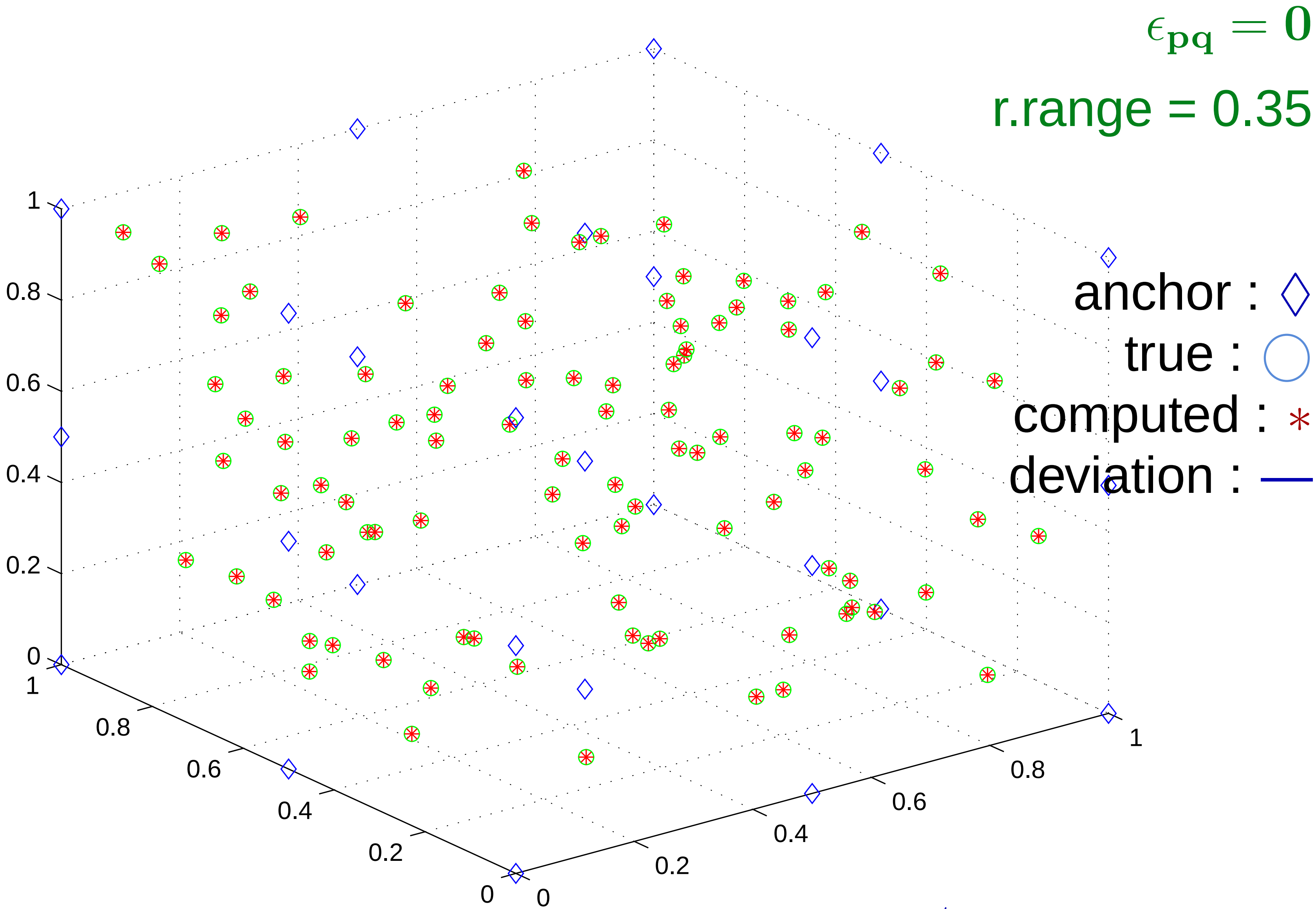
- radio range = 0.25, 0.30
  $\Rightarrow$ Not enough edges to determine all sensors' locations

100 sensors, 27 anchors, $3 \times 3 \times 3$ grid



267.9 sec., rmsd $= \frac{1}{m}\left(\sum_{p=1}^{m} \|\boldsymbol{x}^p - \boldsymbol{a}^p\|^2\right)^{1/2}$ = 3.4e-03

100 sensors, 27 anchors, $3 \times 3 \times 3$ grid

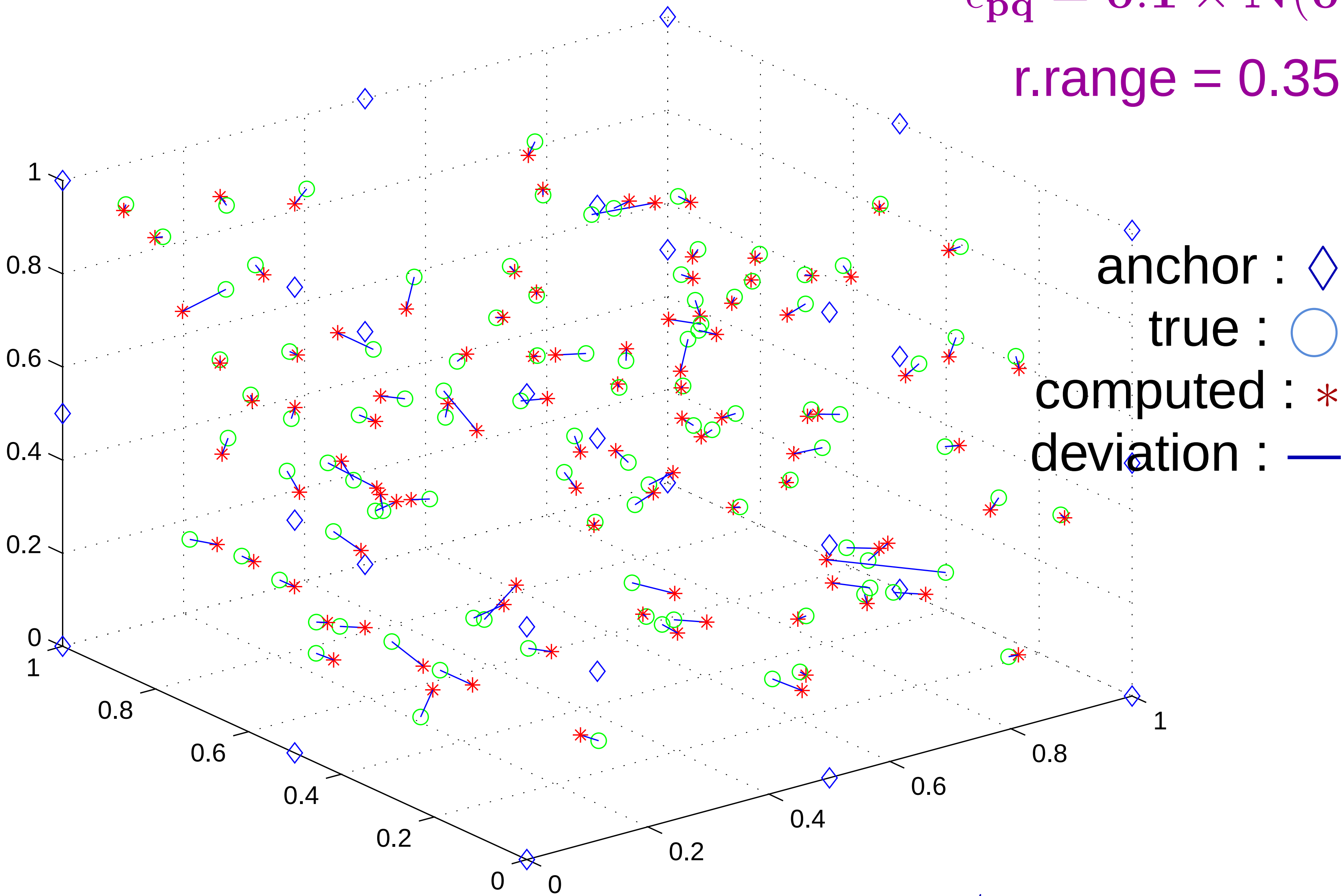


9.9 sec., rmsd $= \frac{1}{m}\left(\sum_{p=1}^{m} \|\boldsymbol{x}^p - \boldsymbol{a}^p\|^2\right)^{1/2}$ = 3.7e-09

100 sensors, 27 anchors, $3 \times 3 \times 3$ grid, $\epsilon_{pq} = 0.1 \times N(0,1)$

$\boldsymbol{\epsilon_{pq} = 0.1 \times N(0,1)}$

r.range = 0.35



14.4 sec., rmsd $= \frac{1}{m}\left(\sum_{p=1}^{m} \|\boldsymbol{x}^p - \boldsymbol{a}^p\|^2\right)^{1/2}$ = 4.5e-03

Contents

Concluding remarks

- Sparse SDP relaxation (Waki-Kim-Kojima-Muramatsu)
  = Lasserre's (dense) SDP relaxation + c-sparsity
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- Thee remain many issues to be studied.
  - Exploiting sparsity further to solve larger scale POPs.
  - Large-scale SDPs.
  - Numerical difficulty in solving SDP relaxations of POPs.
  - Practically effective SDP relaxation for Polynomial SDPs.

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Thank you!