SDP and SOCP relaxations of a Class of Quadratic Optimization Problems

Masakazu Kojima
Tokyo Institute of Technology, Tokyo, Japan
kojima@is.titech.ac.jp

Sunyoung Kim
Ewha Women's University, Seoul, Korea
skim@math.ewha.ac.kr

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This talk

1. **Quadratic Optimization Problem (QOP)**
2. **SDP and Lift-and-Project LP relaxations of QOPs**
3. **SOCP relaxation — 1**
4. **SOCP relaxation — 2**
5. **A Class of QOPs that can be solved by SDP and SOCP**
   $\Longleftarrow$ An extension of S.Zhang 2000
6. **Numerical experiments**
7. **Invariance under Linear Transformation**

1. QOP (Quadratic Optimization Problem)

$$\begin{array}{ll} \textbf{Min.} & \boldsymbol{x}^T \boldsymbol{Q}_0 \boldsymbol{x} + 2\boldsymbol{q}_0^T \boldsymbol{x} + \gamma_0 \\ \textbf{sub.to} & \boldsymbol{x}^T \boldsymbol{Q}_i \boldsymbol{x} + 2\boldsymbol{q}_i^T \boldsymbol{x} + \gamma_i \leq 0 \ (1 \leq i \leq m) \end{array}$$

Here $\boldsymbol{Q}_i \in \mathcal{S}^n$ **(the set of** $n \times n$ **symmetric matrices)**
$\boldsymbol{q}_i \in \mathbb{R}^n$ **(the** n **dimensional Euclidean space)**
$\gamma_i \in \mathbb{R}$ **(the set of real numbers)**

Let $\boldsymbol{M}_i = \begin{pmatrix} \gamma_i & \boldsymbol{q}_i^T \\ \boldsymbol{q}_i & \boldsymbol{Q}_i \end{pmatrix} \in \mathcal{S}^{1+n}$. **Then we can rewrite**

$$\begin{aligned} \boldsymbol{x}^T \boldsymbol{Q}_i \boldsymbol{x} + 2\boldsymbol{q}_i^T \boldsymbol{x} + \gamma_i &\equiv \begin{pmatrix} \gamma_i & \boldsymbol{q}_i^T \\ \boldsymbol{q}_i & \boldsymbol{Q}_i \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{x}\boldsymbol{x}^T \end{pmatrix} \\ &\equiv \boldsymbol{M}_i \bullet \boldsymbol{X} \equiv \sum_{j=0}^{n} \sum_{k=0}^{n} [M_i]_{jk} X_{jk}. \end{aligned}$$

Here

$$\boldsymbol{X} = \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \bar{\boldsymbol{X}} \end{pmatrix} \in \mathcal{S}_+^{1+n}, \ \bar{\boldsymbol{X}} = \boldsymbol{x}\boldsymbol{x}^T.$$

$$\textbf{QOP:} \quad \textbf{min.} \quad \boldsymbol{M}_0 \bullet \boldsymbol{X}$$

$$\textbf{s.t.} \quad \boldsymbol{M}_i \bullet \boldsymbol{X} \leq 0 \ (1 \leq i \leq m), \ \ \boldsymbol{X} = \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \bar{\boldsymbol{X}} \end{pmatrix} \in \mathcal{S}^{1+n}_+, \ \ \bar{\boldsymbol{X}} = \boldsymbol{x}\boldsymbol{x}^T$$

2. SDP and Lift-and-Project LP relaxations of QOPs

SDP Relaxation $\Downarrow$

$$\begin{array}{ll} \textbf{min.} & \boldsymbol{M}_0 \bullet \boldsymbol{X} \\ \textbf{s.t.} & \boldsymbol{M}_i \bullet \boldsymbol{X} \leq 0 \ (1 \leq i \leq m), \ \boldsymbol{X} = \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \bar{\boldsymbol{X}} \end{pmatrix} \in \mathcal{S}^{1+n}_+ \end{array}$$

Lift-and-Project LP Relaxation $\Downarrow$

$$\begin{array}{ll} \textbf{min.} & \boldsymbol{M}_0 \bullet \boldsymbol{X} \\ \textbf{s.t.} & \boldsymbol{M}_i \bullet \boldsymbol{X} \leq 0 \ (1 \leq i \leq m), \ \boldsymbol{X} = \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \bar{\boldsymbol{X}} \end{pmatrix} \in \mathcal{S}^{1+n} \end{array}$$

SDP is more effective than LP but more expensive.
$\Longrightarrow$ **SOCP relaxation as a reasonable compromise.**

3. SOCP Relaxation — 1 (Kim-Kojima 2001).

$$\textbf{QOP:} \quad \textbf{min.} \quad \boldsymbol{Q}_0 \bullet \bar{\boldsymbol{X}} + 2\boldsymbol{q}_0^T \boldsymbol{x} + \gamma_0$$
$$\textbf{s.t.} \quad \boldsymbol{Q}_i \bullet \bar{\boldsymbol{X}} + 2\boldsymbol{q}_i^T \boldsymbol{x} + \gamma_i \leq 0 \ (1 \leq i \leq m), \ \boldsymbol{x}\boldsymbol{x}^T - \bar{\boldsymbol{X}} = \boldsymbol{O}$$

Relaxation $\Downarrow$ $\qquad -(\boldsymbol{x}\boldsymbol{x}^T - \bar{\boldsymbol{X}}) \in \mathcal{S}^n_+$

$$\textbf{min.} \quad \boldsymbol{Q}_0 \bullet \bar{\boldsymbol{X}} + 2\boldsymbol{q}_0^T \boldsymbol{x} + \gamma_0$$
$$\textbf{s.t.} \quad \boldsymbol{Q}_i \bullet \bar{\boldsymbol{X}} + 2\boldsymbol{q}_i^T \boldsymbol{x} + \gamma_i \leq 0 \ (1 \leq i \leq m), \ \boldsymbol{C} \bullet (\boldsymbol{x}\boldsymbol{x}^T - \bar{\boldsymbol{X}}) \leq 0 \ \textbf{for} \ \forall \boldsymbol{C} \in \mathcal{S}^n_+$$

Further Relaxation $\Downarrow$

$$\textbf{SOCP1:} \quad \textbf{min.} \quad \boldsymbol{Q}_0 \bullet \bar{\boldsymbol{X}} + 2\boldsymbol{q}_0^T \boldsymbol{x} + \gamma_0$$
$$\textbf{s.t.} \quad \boldsymbol{Q}_i \bullet \bar{\boldsymbol{X}} + 2\boldsymbol{q}_i^T \boldsymbol{x} + \gamma_i \leq 0 \ (1 \leq i \leq m),$$
$$\boldsymbol{C}_p \bullet (\boldsymbol{x}\boldsymbol{x}^T - \bar{\boldsymbol{X}}) \leq 0 \ (1 \leq p \leq \ell)$$

Here $\boldsymbol{C}_p \in \mathcal{S}^n_+ \ (1 \leq p \leq \ell)$**. Note that each** $\boldsymbol{C}_p \bullet (\boldsymbol{x}\boldsymbol{x}^T - \bar{\boldsymbol{X}}) \leq 0$ **is linear in** $\bar{\boldsymbol{X}}$ **and convex quadratic in** $\boldsymbol{x}$**.**

$\Longrightarrow$ **an SOCP inequality constraint.**

How strong is the inequality $\boldsymbol{C}_p \bullet (\boldsymbol{x}\boldsymbol{x}^T - \bar{\boldsymbol{X}}) \leq 0$?

Comparison:

$$\bar{\boldsymbol{X}} - \boldsymbol{x}\boldsymbol{x}^T \in \mathcal{S}^n_+ \quad \Longleftrightarrow \quad \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \bar{\boldsymbol{X}} \end{pmatrix} \in \mathcal{S}^{1+n}_+$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\boldsymbol{C} \bullet (\bar{\boldsymbol{X}} - \boldsymbol{x}\boldsymbol{x}^T) \geq 0$$

(or $\boldsymbol{C} \bullet (\boldsymbol{x}\boldsymbol{x}^T - \bar{\boldsymbol{X}}) \leq 0$ **)**

$$\begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \bar{\boldsymbol{X}} \end{pmatrix} \bullet \begin{pmatrix} \delta & \boldsymbol{d}^T \\ \boldsymbol{d} & \boldsymbol{C} \end{pmatrix} \geq 0,$$

where $\boldsymbol{C} \in \mathcal{S}^n_+$ **where** $\begin{pmatrix} \delta & \boldsymbol{d}^T \\ \boldsymbol{d} & \boldsymbol{C} \end{pmatrix} \in \mathcal{S}^{1+n}_+$

Which is stronger?

Let $\boldsymbol{C} \in \mathcal{S}^n_+$ be fixed.

$$\boldsymbol{C} \bullet (\bar{\boldsymbol{X}} - \boldsymbol{x}\boldsymbol{x}^T) \geq 0 \Longleftrightarrow \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \bar{\boldsymbol{X}} \end{pmatrix} \bullet \boldsymbol{B} \geq 0 \quad \textbf{for } \forall \boldsymbol{B} \in \mathcal{S}^{1+n}_+ \textbf{ of the form}$$

$$\boldsymbol{B} = \begin{pmatrix} \delta & \boldsymbol{d}^T \\ \boldsymbol{d} & \boldsymbol{C} \end{pmatrix}$$

Conversion of $\boldsymbol{C}_p \bullet (\boldsymbol{x}\boldsymbol{x}^T - \bar{\boldsymbol{X}}) \leq 0$ into an SOCP constraint

Since $\boldsymbol{C}_p$ is positive semidefinite, we can take an $n \times \ell$ matrix $\boldsymbol{L}_p$ such that $\boldsymbol{C}_p = \boldsymbol{L}_p \boldsymbol{L}_p^T$. Then we can rewrite the inequality above as an SOCP constraint:

$$\begin{pmatrix} v_{p0} \\ \boldsymbol{v}_p \end{pmatrix} = \begin{pmatrix} 1 - \boldsymbol{C}_p \bullet \boldsymbol{X} \\ 1 + \boldsymbol{C}_p \bullet \boldsymbol{X} \\ \boldsymbol{L}_p^T \boldsymbol{x} \end{pmatrix},$$

$$\|\boldsymbol{v}_p\| \leq v_{p0}$$

4. SOCP Relaxation — 2

QOP: **min.** $\boldsymbol{M}_0 \bullet \boldsymbol{X}$

s.t. $\boldsymbol{M}_i \bullet \boldsymbol{X} \leq 0 \ (1 \leq i \leq m), \ \boldsymbol{X} = \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \bar{\boldsymbol{X}} \end{pmatrix} \in \mathcal{S}^{1+n}_+, \ \bar{\boldsymbol{X}} = \boldsymbol{x}\boldsymbol{x}^T$

SOCP Relaxation $\Downarrow$

SOCP2: **min.** $\boldsymbol{M}_0 \bullet \boldsymbol{X}$ **s.t.** $\boldsymbol{M}_i \bullet \boldsymbol{X} \leq 0 \ (\forall i), \ X_{00} = 1,$

$\begin{pmatrix} X_{jj} & X_{jk} \\ X_{kj} & X_{kk} \end{pmatrix} \in \mathcal{S}^2_+$ **for** $\forall (j,k) \in \Lambda$

Here $\boldsymbol{M}_i = \begin{pmatrix} \gamma_i & \boldsymbol{q}_i^T \\ \boldsymbol{q}_i & \boldsymbol{Q}_i \end{pmatrix} \in \mathcal{S}^{1+n}$,

$$\boldsymbol{x}^T \boldsymbol{Q}_i \boldsymbol{x} + 2\boldsymbol{q}_i^T \boldsymbol{x} + \gamma_i \equiv \begin{pmatrix} \gamma_i & \boldsymbol{q}_i^T \\ \boldsymbol{q}_i & \boldsymbol{Q}_i \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{x}\boldsymbol{x}^T \end{pmatrix} \equiv \boldsymbol{M}_i \bullet \boldsymbol{X},$$

$\Lambda \equiv \{(j,k) : 0 \leq j < k \leq n,\ [M_i]_{jk} \neq 0$ **for** $\exists i\}$;

$\Lambda^c \equiv \{(j,k) : 0 \leq j < k \leq n,\ [M_i]_{jk} = 0$ **for** $\forall i\}$;

no off-diagonal $X_{jk}\ (j,k) \in \Lambda^c$ **in SOCP2**

$\Rightarrow$ **Advantage when** $\boldsymbol{M}_0, \ldots, \boldsymbol{M}_m$ **are sparse.**

Note that

$$\begin{pmatrix} X_{jj} & X_{jk} \\ X_{kj} & X_{kk} \end{pmatrix} \in \mathcal{S}^n_+ \iff X_{jj} \geq 0,\ X_{kk} \geq 0,\ X_{jj}X_{kk} - X_{jk}^2 \geq 0$$

$$\iff \left\| \begin{pmatrix} X_{jj} - Xkk \\ 2X_{jk} \end{pmatrix} \right\| \leq X_{jj} + X_{kk} \textbf{ (an SOCP constraint)}$$

(a) SOCP relaxation is weaker than SDP relaxation.

(b) SOCP relaxation is stronger than Lift-Project LP relaxation.

(c) A class of QOPs that can be solved by SOCP relaxation?

5. A class of QOPs that can be solved by SOCP

$$\text{QOP:} \quad \min. \quad \boldsymbol{M}_0 \bullet \boldsymbol{X}$$
$$\text{s.t.} \quad \boldsymbol{M}_i \bullet \boldsymbol{X} \leq 0 \ (1 \leq i \leq m), \ \boldsymbol{X} = \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \bar{\boldsymbol{X}} \end{pmatrix} \in \mathcal{S}^{1+n}_+, \ \bar{\boldsymbol{X}} = \boldsymbol{x}\boldsymbol{x}^T$$

ODN-Assumption (OD-non-positiveness): All off-diagonal elements of $\boldsymbol{M}_i$ $(0 \leq i \leq m)$ are non-positive. S.Zhang 2000.

Here $\quad \boldsymbol{M}_i = \begin{pmatrix} \gamma_i & \boldsymbol{q}_i^T \\ \boldsymbol{q}_i & \boldsymbol{Q}_i \end{pmatrix} \in \mathcal{S}^{1+n},$

$$\boldsymbol{x}^T\boldsymbol{Q}_i\boldsymbol{x} + 2\boldsymbol{q}_i^T\boldsymbol{x} + \gamma_i \equiv \begin{pmatrix} \gamma_i & \boldsymbol{q}_i^T \\ \boldsymbol{q}_i & \boldsymbol{Q}_i \end{pmatrix} \bullet \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{x}\boldsymbol{x}^T \end{pmatrix} \equiv \boldsymbol{M}_i \bullet \boldsymbol{X}.$$

Example.

$$
\begin{aligned}
\textbf{Min.} \quad & 6x_1^2 + 3x_2^2 + 4x_3^2 - 2x_1x_2 - 3x_2x_3 - x_1 - 2x_2 \\
\textbf{sub.to} \quad & 2x_1^2 + x_2^2 + 5x_3^2 - 2x_1x_2 - x_2x_3 - 3x_3 \leq 0 \\
& -x_1^2 + 4x_2^2 - 2x_1x_2 - 4x_1x_3 - x_2 \leq 0 \\
& 3x_1^2 + 2x_2^2 - x_3^2 - x_1x_2 - 2x_2x_3 \leq 0
\end{aligned}
$$

SOCP2 (and SDP) relaxation attain the same optimal value as QOP. Let $\boldsymbol{X}$ be an optimal solution of SOCP2 (or SDP). Then $\hat{\boldsymbol{x}} = \left(\sqrt{X_{11}}, \sqrt{X_{22}}, \ldots, \sqrt{X_{nn}}\right)^T$ is an optimal solution of QOP

SOCP2: min. $\boldsymbol{M}_0 \bullet \boldsymbol{X}$ **s.t.** $\boldsymbol{M}_i \bullet \boldsymbol{X} \leq 0 \ (\forall i), \ X_{00} = 1,$

$$\begin{pmatrix} X_{jj} & X_{jk} \\ X_{kj} & X_{kk} \end{pmatrix} \in \mathcal{S}^2_+ \quad \textbf{for } \forall (j,k) \in \Lambda$$

Here $\Lambda = \{(j,k) : 0 \leq j < k \leq n, \ [M_i]_{jk} \neq 0 \ \ \textbf{for } \exists i\}$**,** *i.e.***,**

$\Lambda^c = \{(j,k) : 0 \leq j < k \leq n, \ [M_i]_{jk} = 0 \ \ \textbf{for } \forall i\}$**; no off-diagonal** X_{jk} $(j,k) \in \Lambda^c$ **in SOCP2.** $\Rightarrow$ **Advantage when** $\boldsymbol{M}_0, \ldots, \boldsymbol{M}_m$ **are sparse.**

(i) Different from convexity.

(ii) Some numerical results comparing SOCP and SDP relaxations $\Longrightarrow$ next.

Proof: Let $\boldsymbol{X} = \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{X} \end{pmatrix}$ **be an optimal solution of**

$$\begin{array}{rll} \textbf{SOCP2:} & \textbf{min.} & \boldsymbol{M}_0 \bullet \boldsymbol{X} \\ & \textbf{s.t.} & \boldsymbol{M}_i \bullet \boldsymbol{X} \leq 0 \ (1 \leq i \leq m), \ X_{00} = 1, \\ & & \begin{pmatrix} X_{jj} & X_{jk} \\ X_{kj} & X_{kk} \end{pmatrix} \in \mathcal{S}^2_+ \ \ \textbf{for } \forall (j,k) \in \Lambda \ \ \text{—} \ \textbf{(*)} \end{array}$$

where $\Lambda = \{(j,k) : 0 \leq j < k \leq n, \ [M_i]_{jk} \neq 0 \ \ \textbf{for } \exists i\}$, *i.e.*,
$\Lambda^c = \{(j,k) : 0 \leq j < k \leq n, \ [M_i]_{jk} = 0 \ \ \textbf{for } \forall i\}$. **Let**

$$\hat{\boldsymbol{x}} = \left(\sqrt{X_{11}}, \sqrt{X_{22}}, \ldots, \sqrt{X_{nn}}\right)^T$$

Then $[\boldsymbol{M}_i]_{jj}\hat{x}_j^2 = [\boldsymbol{M}_i]_{jj}X_{jj} \ (0 \leq j \leq n, \ 0 \leq i \leq m)$.

By (*), $0 \leq X_{jj}$, $0 \leq X_{kk}$, $X_{jk}^2 \leq X_{jj}X_{kk} \ ((j,k) \in \Lambda)$,
and by ODN-Assumpt., $[M_i]_{jk} \leq 0 \ ((j,k) \in \Lambda)$; **hence**

$$[M_i]_{jk}\hat{x}_j\hat{x}_k = [\boldsymbol{M}_i]_{jk}\sqrt{X_{jj}}\sqrt{X_{kk}} \leq [\boldsymbol{M}_i]_{jk}X_{jk} \quad ((j,k) \in \Lambda, \ 0 \leq i \leq m).$$

Therefore $\hat{x}$ **satisfies**

$$\sum\sum[M_0]_{jk}\hat{x}_j\hat{x}_k \leq \sum\sum[M_0]_{jk}X_{jk} = \boldsymbol{M}_0 \bullet \boldsymbol{X} \ \textbf{(objective function)},$$
$$\sum\sum[M_i]_{jk}\hat{x}_j\hat{x}_k \leq \boldsymbol{M}_i \bullet \boldsymbol{X} \leq 0 \ (1 \leq p \leq m) \ \ \textbf{(constraint)}.$$

6. Numerical Results comparing SOCP2 and SDP relaxations

$$\begin{array}{rl} \textbf{QOP: min.} & \boldsymbol{x}^T\boldsymbol{Q}_0\boldsymbol{x} + 2\boldsymbol{q}_0^T\boldsymbol{x} + \gamma_0 \\ \textbf{s.t.} & \boldsymbol{x} \in \mathbb{R}^n,\ \boldsymbol{x}^T\boldsymbol{Q}_i\boldsymbol{x} + 2\boldsymbol{q}_i^T\boldsymbol{x} + \gamma_i \leq 0\ (1 \leq i \leq m) \end{array}$$

- **SeDuMi ver.1.03, Sun Enterprise 4500 (CPU 400MHz)**
- **Random sparse matrices $\boldsymbol{Q}_i$ and vectors $\boldsymbol{q}_i$ $(0 \leq i \leq m)$**
- **Each problem generated satisfies ODN-Assump.**
- **Three parameters, n, m and the density of $\boldsymbol{Q}_i$ and $\boldsymbol{q}_i$.**

n	m	**Density**	**SDP**		**SOCP2**		**cpu.ratio**
			cpu	**it.**	**cpu**	**it.**	
200	**100**	**5%**	**198.8**	**19**	**15.1**	**18**	**13.1**
200	**100**	**10%**	**290.4**	**19**	**28.8**	**20**	**10.1**
200	**100**	**50%**	**1430.7**	**31**	**173.1**	**27**	**8.3**
200	**100**	**70%**	**1858.3**	**33**	**212.4**	**26**	**8.7**
200	**100**	**100%**	**2282.8**	**33**	**342.5**	**32**	**6.7**

Table 1. QOPs with $n = 200$**,** $m = 100$ **and varying density**

n	m	Density	SDP		SOCP2		cpu.ratio
			cpu	it.	cpu	it.	
100	**50**	**10%**	**18.3**	**16**	**2.3**	**15**	**8.0**
100	**100**	**10%**	**42.1**	**20**	**6.5**	**19**	**6.5**
100	**200**	**10%**	**125.4**	**18**	**17.7**	**20**	**7.1**
100	**400**	**10%**	**733.1**	**19**	**95.6**	**19**	**7.7**

Table 2. QOPs with $n = 100$ and varying m

n	m	Density	SDP		SOCP2		cpu.ratio
			cpu	it.	cpu	it.	
50	100	10%	12.6	15	1.3	13	9.7
100	100	10%	42.1	20	6.5	19	6.5
200	100	10%	290.4	19	28.8	20	10.1
400	100	10%	3910.4	25	236.7	36	16.5

Table 3. QOPs with $m = 100$ and varying n

Summary: SDP, LP and SOCP2 relaxations

QOP: min. $\boldsymbol{x}^T\boldsymbol{Q}_0\boldsymbol{x} + 2\boldsymbol{q}_0^T\boldsymbol{x} + \gamma_0$
s.t. $\boldsymbol{x}^T\boldsymbol{Q}_i\boldsymbol{x} + 2\boldsymbol{q}_i^T\boldsymbol{x} + \gamma_i \leq 0$

$$\boldsymbol{M}_i = \begin{pmatrix} \gamma_i & \boldsymbol{q}_i^T \\ \boldsymbol{q}_i & \boldsymbol{Q}_i \end{pmatrix} \quad (0 \leq i \leq m),$$

QOP: min. $\boldsymbol{M}_0 \bullet \boldsymbol{X}$
s.t. $\boldsymbol{M}_i \bullet \boldsymbol{X} \leq 0 \; (1 \leq i \leq m), \; X_{00} = 1$ — **(♯)**,

$$\boldsymbol{X} = \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \bar{\boldsymbol{X}} \end{pmatrix} \in \mathcal{S}_+^{1+n}, \; \bar{\boldsymbol{X}} = \boldsymbol{x}\boldsymbol{x}^T$$

SDP: min. $\boldsymbol{M}_0 \bullet \boldsymbol{X}$ s.t. (♯) and $\boldsymbol{X} \in \mathcal{S}_+^{1+n}$

SOCP2: min. $\boldsymbol{M}_0 \bullet \boldsymbol{X}$ s.t. (♯) and $\begin{pmatrix} X_{jj} & X_{jk} \\ X_{kj} & X_{kk} \end{pmatrix} \in \mathcal{S}_+^2$ for $\forall (j,k) \in \Lambda$

LP: min. $\boldsymbol{M}_0 \bullet \boldsymbol{X}$ s.t. (♯)

Here $\Lambda = \{(j,k) : 0 \leq j < k \leq n, \; [M_i]_{jk} \neq 0$ **for** $\exists i\}$, *i.e.*,
$\Lambda^c = \{(j,k) : 0 \leq j < k \leq n, \; [M_i]_{jk} = 0$ **for** $\forall i\}$.

- **Quality of obj. values: QOP $\leq$ SDP $\leq$ SOCP2 $\leq$ LP in general.**
  QOP $=$ SDP $=$ SOCP2 $\leq$ LP under ODN-Assump.
- **CPU time to solve the problems: SDP $\geq$ SOCP2 $\geq$ LP.**
- **Density: no off-diagonal X_{jk} $(j,k) \in \Lambda^c$ in QOP, SOCP2, LP.**

7. Invariance under linear transformation

Let $\boldsymbol{P}$ be an $n \times n$ nonsing. matrix. Apply a linear trans. $\boldsymbol{x} = \bar{\boldsymbol{P}}\boldsymbol{y}$ to

QOP: min. $\boldsymbol{M}_0 \bullet \boldsymbol{X}$

s.t. $\boldsymbol{M}_i \bullet \boldsymbol{X} \leq 0 \ (1 \leq i \leq m), \ \boldsymbol{X} = \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \bar{\boldsymbol{X}} \end{pmatrix} \in \mathcal{S}^{1+n}_+, \ \bar{\boldsymbol{X}} = \boldsymbol{x}\boldsymbol{x}^T$

to obtain an equiv. QOP' in the matrix variable

$$\boldsymbol{Y} = \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \bar{\boldsymbol{Y}} \end{pmatrix} = \boldsymbol{P}\boldsymbol{X}\boldsymbol{P}^T = \begin{pmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{0} & \bar{\boldsymbol{P}} \end{pmatrix} \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \bar{\boldsymbol{X}} \end{pmatrix} \begin{pmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{0} & \bar{\boldsymbol{P}}^T \end{pmatrix}, \ \bar{\boldsymbol{Y}} = \boldsymbol{y}\boldsymbol{y}^T$$

with the data matrices $\boldsymbol{M}'_i = \boldsymbol{P}\boldsymbol{M}_i\boldsymbol{P}^T$.

$$\textbf{QOP': min. } \boldsymbol{M}_0' \bullet \boldsymbol{Y}$$

$$\textbf{s.t. } \boldsymbol{M}_i' \bullet \boldsymbol{Y} \leq 0 \ (1 \leq i \leq m), \ \boldsymbol{Y} = \begin{pmatrix} 1 & \boldsymbol{y}^T \\ \boldsymbol{y} & \bar{\boldsymbol{Y}} \end{pmatrix} \in \mathcal{S}_+^{1+n}, \ \bar{\boldsymbol{Y}} = \boldsymbol{y}\boldsymbol{y}^T$$

$$
\begin{array}{cccc}
 & \textbf{QOP} & \Longrightarrow & \textbf{SDP} \\
\boldsymbol{M}_i' = \boldsymbol{P}\boldsymbol{M}_i\boldsymbol{P}^T & \Updownarrow & & \Updownarrow \\
 & \textbf{QOP'} & \Longrightarrow & \textbf{SDP'} \\
 & & \textbf{SDP relaxation} &
\end{array}
$$

$$
\begin{array}{cccc}
 & \textbf{QOP} & \Longrightarrow & \textbf{SOCP2} \\
\boldsymbol{M}'_i = \boldsymbol{P}\boldsymbol{M}_i\boldsymbol{P}^T & \Updownarrow & & \Updownarrow \\
 & \textbf{QOP'} & \Longrightarrow & \textbf{SOCP2'}
\end{array}
$$

SOCP relaxation — 2

$\Updownarrow$ **: Valid when $\bar{P}$ is diag. or permut. mat. but invalid in general.**

IODN-Assumpt. (Implicit OD-non-positiveness): $\exists$ nonsig. $\bar{\boldsymbol{P}}$ (unknown); all off-diagonal elements of $\boldsymbol{M}'_i$, $\forall i$ are non-positive.

(i) Opt. val. : SOCP2 "=" SDP = QOP, where "=" holds when $\bar{P}$ is a diag. or permut. mat. but not in general.

(ii) Opt. sol.: "If $\boldsymbol{X}$ is an opt. sol. of SOCP2 (or SDP), then $\hat{\boldsymbol{x}} = \left(\sqrt{X_{11}}, \ldots, \sqrt{X_{nn}}\right)^T$ is an opt. sol. of QOP" does not hold.

- **Can we verify whether a given QOP satisfies IODN-Assump?**
- **How we construct an opt. sol. of a QOP satisfying IODN-Assumpt?**

Concluding Remarks

(a) Two types of SOCP relaxations, SOCP1 and SOCP2.

(b) Reasonable compromise between the effectiveness of the SDP relaxation and the low computational cost of the lift-and-project LP relaxation.

(c) A class of QOPs that can be solved by the SOCP2 — ODN-Assumption.

(d) Application? — Should be combinedly used with the branch and bound method for solving general QOPs; the role of SOCP relaxations is to compute effective bounds for objective values for subproblems at low cost.