A Numerical Algorithm for Block-Diagonal Decomposition of Matrix *-Algebra

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Given $n \times n$ symmetric matrices $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m$, find an orthogonal matrix $\boldsymbol{P}$ which block-diagonalizes them simultaneously.

$$\boldsymbol{A}_1 = \begin{pmatrix} 2.66\ldots & -0.94\ldots & 0 \\ -0.94\ldots & -1.16\ldots & -2.59\ldots \\ 0 & -2.59\ldots & 0.50\ldots \end{pmatrix},$$

$$\boldsymbol{A}_2 = \begin{pmatrix} 3.00\ldots & 0 & 0 \\ 0 & 1.50\ldots & -0.86\ldots \\ 0 & -0.86\ldots & 2.50\ldots \end{pmatrix},$$

$$\boldsymbol{A}_3 = \begin{pmatrix} 0 & -0.70\ldots & -0.40\ldots \\ -0.70\ldots & 3.50\ldots & 1.44\ldots \\ -0.40\ldots & 1.44\ldots & 0.50\ldots \end{pmatrix}.$$

Let $\boldsymbol{P} = \begin{pmatrix} 0.57\ldots & 0.44\ldots & -0.68\ldots \\ 0.40\ldots & 0.56\ldots & 0.71\ldots \\ -0.70\ldots & 0.69\ldots & -0.14\ldots \end{pmatrix}$. Then

Given $n \times n$ symmetric matrices $\boldsymbol{A}_1, \dots, \boldsymbol{A}_m$, find an orthogonal matrix $\boldsymbol{P}$ which block-diagonalizes them simultaneously.

$$\boldsymbol{P}^T \boldsymbol{A}_1 \boldsymbol{P} = \begin{pmatrix} 2.00\dots & 0 & 0 \\ 0 & -2.09\dots & -2.36\dots \\ 0 & -2.36\dots & 2.09\dots \end{pmatrix},$$

$$\boldsymbol{P}^T \boldsymbol{A}_2 \boldsymbol{P} = \begin{pmatrix} 3.00\dots & 0 & 0 \\ 0 & 1.60\dots & -0.91\dots \\ 0 & -0.91\dots & 2.39\dots \end{pmatrix},$$

$$\boldsymbol{P}^T \boldsymbol{A}_3 \boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1.86\dots & 2.23\dots \\ 0 & 2.23\dots & 2.13\dots \end{pmatrix}.$$

- How do we compute such a $\boldsymbol{P}$ using only numerical values of $\boldsymbol{A}_1, \dots, \boldsymbol{A}_m$?
- $\boldsymbol{A}_1, \dots, \boldsymbol{A}_m$: data matrices of an SDP
  $\Rightarrow$ an SDP with blockdiagonal data matrices

Given $n \times n$ symmetric matrices $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m$, find an orthogonal matrix $\boldsymbol{P}$ which block-diagonalizes them simultaneously.

Outline

1. Theoretical Framework
2. Numerical Method
3. Some Examples
4. Concluding Remarks

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Theoretical frameworks closely related

(a) Group representation theory
(b) The theory of matrix *-algebra

These frameworks are used in existing methods

[1] Bai-de Klerk-Pasechnik-Sotirov 2007 — truss topology optimization $\Rightarrow$ SDP
[2] de Klerk-Pasechnik-Schrijver 2007 — SDP
[3] de Klerk-Sotirov 2007 — quadratic assignment problem
[4] Gatermann-Parrilo 2004 — SDP and sum of squares
[5] Jansson-Lasserre-Riener-Thebald 2006 — SDP relaxation of POP
[6] Kanno-Ohsaki-Murota-Katoh 2001 — truss topology optimization $\Rightarrow$ SDP

Theoretical frameworks closely related

(a) Group representation theory

(b) The theory of matrix *-algebra

- An algebraic structure such as group symmetry and matrix *-algebra behind a class of problems is assumed to be known in advance.
- However, a given problem is a specific instance in the class, so it may satisfy an additional algebraic structure (often induced from sparsity).

Our method assumes no knowledge on the algebraic structure of a given problem in advance.

- Eberly and Giesbrecht, “Efficient decomposition of separable algebra”, J. Sym. Comp., 2004.

matrix *-algebra	$\subset$	seprable algebra
orthog. mat.,	vs	nonsing. mat.
Phase 1 and Phase 2	vs	Phase 1

Notation and Definitions

- $\mathcal{M}_n, \mathcal{S}_n$: the sets of $n \times n$ real mat. and real sym. mat.
- $\mathcal{T} \subseteq \mathcal{M}_n$: a *-algebra (or a matrix *-algebra) if $\boldsymbol{I}_n \in \mathcal{T}$ and
$$\boldsymbol{A}, \boldsymbol{B} \in \mathcal{T},\ \alpha, \beta \in \mathbb{R} \Rightarrow \alpha\boldsymbol{A} + \beta\boldsymbol{B},\ \boldsymbol{A}\boldsymbol{B},\ \boldsymbol{A}^T \in \mathcal{T}.$$
- $\bigoplus_{j=1}^{\ell} \boldsymbol{C}_j = \text{diag}\,(\boldsymbol{C}_1, \ldots, \boldsymbol{C}_\ell) = \begin{pmatrix} \boldsymbol{C}_1 & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{C}_2 & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{C}_\ell \end{pmatrix}$

Let $\mathcal{T}$ be a *-algebra generated by $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m \in \mathcal{S}_n$. Then an orthog. mat. $\boldsymbol{P}$ block-diagonalizes $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m \in \mathcal{S}_n$ iff it block-diagonalizes all matrices in $\mathcal{T}$.

a *-algebra generated by symmetric matrices $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m \in \mathcal{S}_n$

Fact 1: Let $\mathcal{T}$ be a *-algebra of $\mathcal{M}_n$. Then $\exists$ orthog. $\widehat{\boldsymbol{Q}}$ and $\exists$ simple *-algebra $\mathcal{T}_j$ $(j = 1, \ldots, \ell)$; $\forall \boldsymbol{A} \in \mathcal{T}$ can be transformed to a block-diagonal form as

$$\widehat{\boldsymbol{Q}}^T \boldsymbol{A} \widehat{\boldsymbol{Q}} = \bigoplus_{j=1}^{\ell} \boldsymbol{C}_j = \text{diag}\,(\boldsymbol{C}_1, \ldots, \boldsymbol{C}_\ell) \text{ for } \exists C_j \in \mathcal{T}_j.$$

- $\mathcal{I} \subseteq \mathcal{T}$: an ideal of $\mathcal{T}$ if $\boldsymbol{A} \in \mathcal{T},\ \boldsymbol{B} \in \mathcal{I} \Rightarrow \boldsymbol{AB} \in \mathcal{I}$.
- $\mathcal{T}$: simple if $\mathcal{T}$ has no ideal other than $\{\boldsymbol{O}\}$ and $\mathcal{T}$ itself.

Examples:

$\mathcal{T} = \mathcal{M}_n$: simple.

$\mathcal{T} = \{\text{diag}(\boldsymbol{B}, \boldsymbol{B}) : \boldsymbol{B} \in \mathcal{M}_n\}$: simple,
which can be decomposed further.

$\mathcal{T} = \{\text{diag}(\boldsymbol{B}_1, \boldsymbol{B}_2) : \boldsymbol{B}_1 \in \mathcal{M}_m,\ \boldsymbol{B}_2 \in \mathcal{M}_m\}$: not simple
because $\mathcal{I} = \{\text{diag}(\boldsymbol{B}_1, \boldsymbol{O}) : \boldsymbol{B}_1 \in \mathcal{M}_m\}$: an ideal of $\mathcal{T}$.

Fact 1: Let $\mathcal{T}$ be a *-algebra of $\mathcal{M}_n$. Then $\exists$ orthog. $\widehat{\boldsymbol{Q}}$ and $\exists$ simple *-algebra $\mathcal{T}_j$ $(j = 1, \ldots, \ell)$; $\forall \boldsymbol{A} \in \mathcal{T}$ can be transformed to a block-diagonal form as

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- $\mathcal{T}$: simple if $\mathcal{T}$ has no ideal other than $\{\boldsymbol{O}\}$ and $\mathcal{T}$ itself.

Fact 2: Let $\mathcal{T}$ be a simple *-algebra of $\mathcal{M}_n$. Then

$$\overline{\boldsymbol{Q}}^T \mathcal{T} \overline{\boldsymbol{Q}} = \{\text{diag}\,(\underbrace{\boldsymbol{B}, \ldots, \boldsymbol{B}}_{m\text{-tuple}}) : \boldsymbol{B} \in \mathcal{T}'\}$$

for $\exists$ orthog. $\overline{\boldsymbol{Q}}$ and $\exists$ irreducible *-algebra $\mathcal{T}'$.

- a subsp. W of $\mathbb{R}^n$ is $\mathcal{T}$-invariant if $\boldsymbol{A}W \subseteq W$ for $\forall \boldsymbol{A} \in \mathcal{T}$.
- $\mathcal{T}$: irreducible $\Leftrightarrow \nexists$ $\mathcal{T}$-invariant W other than $\{\boldsymbol{0}\}$ and $\mathbb{R}^n$.

Fact 3: Let $\mathcal{T}$ be a irreducible *-algebra of $\mathcal{M}_n$. Then

$$\mathcal{T} \;=\; \mathcal{M}_n = \mathcal{M}_n(\mathbb{R}),\ \mathcal{M}_{n/2}(\mathbb{C}) \text{ or } \mathcal{M}_{n/4}(\mathbb{H}).$$

Here

$\mathcal{M}_n(\mathbb{R})$ = the matrix algebra of $n \times n$ real matrices.

$\mathcal{M}_{n/2}(\mathbb{C})$ = "a real representation of the $(n/2 \times n/2)$ complex matrix algebra"

$\mathcal{M}_{n/4}(\mathbb{H})$ = "$\cdots\cdots$ the $(n/4 \times n/4)$ quaternion mat. algebra"

- ~~When the *-algebra $\mathcal{T}$ is generated by symmetric matrices, $\mathcal{M}_{n/2}(\mathbb{C})$ and $\mathcal{M}_{n/4}(\mathbb{H})$ never occur.~~
- $\mathcal{M}_{n/2}(\mathbb{C})$, $\mathcal{M}_{n/4}(\mathbb{H})$ occur; $\begin{pmatrix} a & 0 & b & c \\ 0 & a & -c & b \\ b & -c & d & 0 \\ c & b & 0 & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{C})$
- Optimization problem from "Schwedler Dome" trusses — an SDP example for $\mathcal{M}_{n/2}(\mathbb{C})$.

Outline

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Let $\mathcal{T}$ be a *-algebra of $\mathcal{M}_n$ generated by sym. matrices $\boldsymbol{I}, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_m$. Then $\exists$ orthog. mat. $\boldsymbol{P}$;

$$\boldsymbol{P}^T \boldsymbol{A} \boldsymbol{P} = \bigoplus_{j=1}^{\ell} \boldsymbol{C}_j = \bigoplus_{j=1}^{\ell} \text{diag}\,(\boldsymbol{B}_j, \ldots, \boldsymbol{B}_j), \ \forall \boldsymbol{A} \in \mathcal{T}.$$

- $\mathcal{T}_j = \{\text{diag}\,(\boldsymbol{B}_j, \ldots, \boldsymbol{B}_j) : \boldsymbol{B}_j \in \mathcal{T}'_j\}$: a simple *-algebra.
- $\mathcal{T}'_j$ is irreducible, $\mathcal{T}'_j = \mathcal{M}_{n_j}(\mathbb{R})$, $\mathcal{M}_{n_j/2}(\mathbb{C})$ or $\mathcal{M}_{n_j/2}(\mathbb{H})$.

A proposed method consists of two phases

Phase 1: Decompose $\mathcal{T}$ into simple $\mathcal{T}_j$ $(j = 1, \ldots, \ell)$.

Phase 2: Decompose $\forall$ simple $\mathcal{T}_j$ into irreducible $\mathcal{T}'_j$s.

Only $\mathcal{M}_{n_j}(\mathbb{R})$ case in this talk.

$\mathcal{M}_{n_j/2}(\mathbb{C})$ and $\mathcal{M}_{n_j/4}(\mathbb{H})$ cases — not studied completely yet.

- Maehara-Murota (June 2008) proposes a numerical method for block-diagonalizing (not necessarily symmetric) $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m \in \mathcal{M}_n$; the method proposed there can process $\mathcal{M}_{n_j/2}(\mathbb{C})$ and $\mathcal{M}_{n_j/4}(\mathbb{H})$ cases.

Let $\mathcal{T}$ be a *-algebra of $\mathcal{M}_n$ generated by sym. matrices $\boldsymbol{I}, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_m$. Then $\exists$ orthog. mat. $\boldsymbol{P}$;

$$\boldsymbol{P}^T \boldsymbol{A} \boldsymbol{P} = \bigoplus_{j=1}^{\ell} \boldsymbol{C}_j = \bigoplus_{j=1}^{\ell} \text{diag}\,(\boldsymbol{B}_j, \ldots, \boldsymbol{B}_j), \ \forall \boldsymbol{A} \in \mathcal{T}.$$

- $\mathcal{T}_j = \{\text{diag}\,(\boldsymbol{B}_j, \ldots, \boldsymbol{B}_j) : \boldsymbol{B}_j \in \mathcal{T}'_j\}$: a simple *-algebra.
- $\mathcal{T}'_j$ is irreducible, $\mathcal{T}'_j = \mathcal{M}_{n_j}(\mathbb{R}),\ \mathcal{M}_{n_j/2}(\mathbb{C})$ or $\mathcal{M}_{n_j/2}(\mathbb{H})$.
- Each $\mathcal{T}_j$ or the image of the corresponding columns of $\boldsymbol{P}$ induces an $\mathcal{T}$-invariant subspace U_j; $\mathbb{R}^n = \bigoplus_{j=1}^{\ell} U_j$.
- Each U_j is decomposed such that $U_j = \bigoplus_{i=1}^{m_j} V_{ji}$ corresponding to the decomposition of $\mathcal{T}_j$.
- Finding $\boldsymbol{P} \Leftrightarrow$ Finding $\mathcal{T}$-invariant spaces U_j, V_{ji}.

Recall that U : $\mathcal{T}$-invariant $\Leftrightarrow \boldsymbol{A}U \subset U$ for $\forall \boldsymbol{A} \in \mathcal{T}$.
$U = \{\boldsymbol{u} : \boldsymbol{A}\boldsymbol{u} = \lambda \boldsymbol{u}\}$ for some eigenvalue of $\boldsymbol{A}$.

- Eigenvalues and eigenvectors of $\boldsymbol{A} \in \mathcal{T}$ contain information on the decomposition above.

Let $\mathcal{T}$ be a *-algebra of $\mathcal{M}_n$ generated by sym. matrices $\boldsymbol{I}, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_m$. Then $\exists$ orthog. mat. $\boldsymbol{P}$;

$$\boldsymbol{P}^T \boldsymbol{A} \boldsymbol{P} = \bigoplus_{j=1}^{\ell} \boldsymbol{C}_j = \bigoplus_{j=1}^{\ell} \text{diag}\,(\boldsymbol{B}_j, \ldots, \boldsymbol{B}_j), \ \forall \boldsymbol{A} \in \mathcal{T}.$$

- $\mathcal{T}_j = \{\text{diag}\,(\boldsymbol{B}_j, \ldots, \boldsymbol{B}_j) : \boldsymbol{B}_j \in \mathcal{T}'_j\}$: a simple *-algebra.
- $\mathcal{T}'_j$ is irreducible, $\mathcal{T}'_j = \mathcal{M}_{n_j}(\mathbb{R}),\ \mathcal{M}_{n_j/2}(\mathbb{C})$ or $\mathcal{M}_{n_j/2}(\mathbb{H})$.

Definition (Valid only for $\mathcal{T}'_j = \mathcal{M}_{n_j}$): $\boldsymbol{A} \in \mathcal{T}$ is generic (in eigenvalue structure) if no two of $\boldsymbol{C}_j$ $(j = 1, \ldots, \ell)$ share a common eigenvalue and $\forall \boldsymbol{B}_j$ has no multiple eigenvalue.

- This definition does not depend on the choice of $\boldsymbol{P}$.
- $\{\boldsymbol{A} \in \mathcal{T} : \boldsymbol{A}$ is generic $\}$ is open and dense w.r.t. $\mathcal{T}$.
- Each generic $\boldsymbol{A}$ involves information on the decompositions $\mathbb{R}^n = \bigoplus_{j=1}^{\ell} U_j$ and $U_j = \bigoplus_{i=1}^{\ell} V_{ji}$ $(j = 1, \ldots, \ell)$.

Let $\mathcal{T}$ be a *-algebra of $\mathcal{M}_n$ generated by sym. matrices $\boldsymbol{I}, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_m$. Then $\exists$ orthog. mat. $\boldsymbol{P}$;

$$\boldsymbol{P}^T \boldsymbol{A} \boldsymbol{P} = \bigoplus_{j=1}^{\ell} \boldsymbol{C}_j = \bigoplus_{j=1}^{\ell} \text{diag}\,(\boldsymbol{B}_j, \ldots, \boldsymbol{B}_j), \ \forall \boldsymbol{A} \in \mathcal{T}.$$

- $\mathcal{T}_j = \{\text{diag}\,(\boldsymbol{B}_j, \ldots, \boldsymbol{B}_j) : \boldsymbol{B}_j \in \mathcal{T}'_j\}$: a simple *-algebra.
- $\mathcal{T}'_j$ is irreducible, $\mathcal{T}'_j = \mathcal{M}_{n_j}(\mathbb{R})$, $\mathcal{M}_{n_j/2}(\mathbb{C})$ or $\mathcal{M}_{n_j/2}(\mathbb{H})$.

1-1. Let $\boldsymbol{A} = \boldsymbol{I} + \sum_{p=1}^{m} r_p \boldsymbol{A}_p$,
where $r_p \in (0, 1)$: random (we expect "$\boldsymbol{A}$ is generic").
1-2. Diagonalize $\boldsymbol{A}$; $\widetilde{\boldsymbol{Q}}^T \boldsymbol{A} \widetilde{\boldsymbol{Q}} = \text{diag}\,(\alpha_1 \boldsymbol{I}_{m_1}, \ldots, \alpha_k \boldsymbol{I}_{m_k})$
(k blocks)

Let $W_i = \{\boldsymbol{v} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{v} = \alpha_i \boldsymbol{v}\}$: the space of eigenvectors assoc. with α_i. Then $W_i \subseteq \exists U_j$; because no two of $\boldsymbol{C}_1, \ldots, \boldsymbol{C}_\ell$ share a common eigenvalue. Here $\mathbb{R}^n = \bigcup_{j=1}^{\ell} U_j$.

Let $\mathcal{T}$ be a *-algebra of $\mathcal{M}_n$ generated by sym. matrices $\boldsymbol{I}, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_m$. Then $\exists$ orthog. mat. $\boldsymbol{P}$;

$$\boldsymbol{P}^T \boldsymbol{A} \boldsymbol{P} = \bigoplus_{j=1}^{\ell} \boldsymbol{C}_j = \bigoplus_{j=1}^{\ell} \text{diag}\,(\boldsymbol{B}_j, \ldots, \boldsymbol{B}_j), \ \forall \boldsymbol{A} \in \mathcal{T}.$$

- $\mathcal{T}_j = \{\text{diag}\,(\boldsymbol{B}_j, \ldots, \boldsymbol{B}_j) : \boldsymbol{B}_j \in \mathcal{T}'_j\}$: a simple *-algebra.
- $\mathcal{T}'_j$ is irreducible, $\mathcal{T}'_j = \mathcal{M}_{n_j}(\mathbb{R})$, $\mathcal{M}_{n_j/2}(\mathbb{C})$ or $\mathcal{M}_{n_j/2}(\mathbb{H})$.

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(k blocks)

1-3. Let $\boldsymbol{A}(\boldsymbol{s}) = \boldsymbol{I} + \sum_{p=1}^{m} s_p \boldsymbol{A}_p$, where $s_p \in (0, 1)$: random.

1-4. Permute k blocks; $\hat{\Pi}^T \widetilde{\boldsymbol{Q}}^T \boldsymbol{A}(\boldsymbol{s}) \widetilde{\boldsymbol{Q}} \hat{\Pi} = \bigoplus_{j=1}^{\ell} \tilde{\boldsymbol{C}}_j$.

2. Block diagonalize $\tilde{\boldsymbol{C}}_j$; $\bar{\boldsymbol{Q}}_j^T \tilde{\boldsymbol{C}}_j \bar{\boldsymbol{Q}}_j = \text{diag}\,(\tilde{\boldsymbol{B}}_j, \ldots, \tilde{\boldsymbol{B}}_j)$.

Let $\boldsymbol{P} = \widetilde{\boldsymbol{Q}} \hat{\Pi} \text{diag}\,(\bar{\boldsymbol{Q}}_1, \ldots, \bar{\boldsymbol{Q}}_\ell)$

If $\neg$"$\boldsymbol{A}$ is generic", we may have (a) $\exists$ inconsistency at 1-4 and/or 2 or (b) the finest decomposition is not attained — later.

Example

1-1. Let $\boldsymbol{A} = \boldsymbol{I} + \sum_{p=1}^{m} r_p \boldsymbol{A}_p$. 1-2. Diagonalize $\boldsymbol{A}$:

$$\widetilde{\boldsymbol{Q}}^T \boldsymbol{A} \widetilde{\boldsymbol{Q}} = \begin{pmatrix} \alpha_1 \boldsymbol{I}_2 & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \alpha_2 \boldsymbol{I}_2 & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \alpha_3 \boldsymbol{I}_3 & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \alpha_4 \boldsymbol{I}_2 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \alpha_5 \boldsymbol{I}_3 \end{pmatrix}$$

1-3. Let $\boldsymbol{A}(\boldsymbol{s}) = \boldsymbol{I} + \sum_{p=1}^{m} s_p \boldsymbol{A}_p$, and compute $\widetilde{\boldsymbol{Q}}^T \boldsymbol{A}(\boldsymbol{s}) \widetilde{\boldsymbol{Q}}$:

$$\widetilde{\boldsymbol{Q}}^T \boldsymbol{A}(\boldsymbol{s}) \widetilde{\boldsymbol{Q}} = \begin{pmatrix} \star_2 & \star_2 & \boldsymbol{O} & \star_2 & \boldsymbol{O} \\ \star_2 & \star_2 & \boldsymbol{O} & \star_2 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \star_3 & \boldsymbol{O} & \star_3 \\ \star_2 & \star_2 & \boldsymbol{O} & \star & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \star_3 & \boldsymbol{O} & \star_3 \end{pmatrix}$$

1-4. Permute 5×5 blocks to have a block diagonal matrix:

$$\widetilde{\Pi}^T \widetilde{\boldsymbol{Q}}^T \boldsymbol{A}(\boldsymbol{s}) \widetilde{\boldsymbol{Q}} \widetilde{\Pi} = \begin{pmatrix} \widetilde{\boldsymbol{A}}_{11} & \widetilde{\boldsymbol{A}}_{12} & \widetilde{\boldsymbol{A}}_{13} & \boldsymbol{O} & \boldsymbol{O} \\ \widetilde{\boldsymbol{A}}_{21} & \widetilde{\boldsymbol{A}}_{22} & \widetilde{\boldsymbol{A}}_{23} & \boldsymbol{O} & \boldsymbol{O} \\ \widetilde{\boldsymbol{A}}_{31} & \widetilde{\boldsymbol{A}}_{32} & \widetilde{\boldsymbol{A}}_{33} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \widetilde{\boldsymbol{A}}_{44} & \widetilde{\boldsymbol{A}}_{45} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \widetilde{\boldsymbol{A}}_{54} & \widetilde{\boldsymbol{A}}_{55} \end{pmatrix}$$

Let $\widehat{\boldsymbol{Q}} = \widetilde{\boldsymbol{Q}} \widetilde{\Pi}$, which block diagonalize all $\boldsymbol{A} \in \mathcal{T}$;

a simple *-algebra $\mathcal{T}_1$

$$\widehat{\boldsymbol{Q}}^T \boldsymbol{A} \widehat{\boldsymbol{Q}} = \begin{pmatrix} \star & \star & \star & \boldsymbol{O} & \boldsymbol{O} \\ \star & \star & \star & \boldsymbol{O} & \boldsymbol{O} \\ \star & \star & \star & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \star & \star \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \star & \star \end{pmatrix}$$

a simple *-algebra $\mathcal{T}_2$ $\swarrow$

Phase 1 is completed.

$\Rightarrow$ Phase 2: Decomposition of $\mathcal{T}_1$ and $\mathcal{T}_2$

$$\widetilde{\Pi}^T \widetilde{\boldsymbol{Q}}^T \boldsymbol{A}(\boldsymbol{s}) \widetilde{\boldsymbol{Q}} \widetilde{\Pi} = \begin{pmatrix} \widetilde{\boldsymbol{A}}_{11} & \widetilde{\boldsymbol{A}}_{12} & \widetilde{\boldsymbol{A}}_{13} & \boldsymbol{O} & \boldsymbol{O} \\ \widetilde{\boldsymbol{A}}_{21} & \widetilde{\boldsymbol{A}}_{22} & \widetilde{\boldsymbol{A}}_{23} & \boldsymbol{O} & \boldsymbol{O} \\ \widetilde{\boldsymbol{A}}_{31} & \widetilde{\boldsymbol{A}}_{32} & \widetilde{\boldsymbol{A}}_{33} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \widetilde{\boldsymbol{A}}_{44} & \widetilde{\boldsymbol{A}}_{45} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \widetilde{\boldsymbol{A}}_{54} & \widetilde{\boldsymbol{A}}_{55} \end{pmatrix}$$

2-1. Solve the following matrix equations in $\bar{\boldsymbol{Q}}_i$ and b_{ij}:

$$\text{diag}\,(\bar{\boldsymbol{Q}}_1\ \bar{\boldsymbol{Q}}_2\ \bar{\boldsymbol{Q}}_3)^T \begin{pmatrix} \widetilde{\boldsymbol{A}}_{11} & \widetilde{\boldsymbol{A}}_{12} & \widetilde{\boldsymbol{A}}_{13} \\ \widetilde{\boldsymbol{A}}_{21} & \widetilde{\boldsymbol{A}}_{22} & \widetilde{\boldsymbol{A}}_{23} \\ \widetilde{\boldsymbol{A}}_{31} & \widetilde{\boldsymbol{A}}_{32} & \widetilde{\boldsymbol{A}}_{33} \end{pmatrix} \text{diag}\,(\bar{\boldsymbol{Q}}_1\ \bar{\boldsymbol{Q}}_2\ \bar{\boldsymbol{Q}}_3)$$

$$= \begin{pmatrix} b_{11}\boldsymbol{I}_2 & b_{12}\boldsymbol{I}_2 & b_{13}\boldsymbol{I}_2 \\ b_{21}\boldsymbol{I}_2 & b_{22}\boldsymbol{I}_2 & b_{23}\boldsymbol{I}_2 \\ b_{31}\boldsymbol{I}_2 & b_{32}\boldsymbol{I}_2 & b_{33}\boldsymbol{I}_2 \end{pmatrix} \text{ with } \bar{\boldsymbol{Q}}_1 = \boldsymbol{I}_2,\ \bar{\boldsymbol{Q}}_j :\ \text{orthogonal.}$$

Consistent under "$\boldsymbol{A}$ and $\boldsymbol{A}(s)$ are generic". Here $\boldsymbol{I}_2 : 2 \times 2$.

2-2. Apply row and column permutation $\bar{\Pi}$:

$$\bar{\Pi}_1^T \text{diag}\,(\bar{\boldsymbol{Q}}_1\ \bar{\boldsymbol{Q}}_2\ \bar{\boldsymbol{Q}}_3)^T \begin{pmatrix} \widetilde{\boldsymbol{A}}_{11} & \widetilde{\boldsymbol{A}}_{12} & \widetilde{\boldsymbol{A}}_{13} \\ \widetilde{\boldsymbol{A}}_{21} & \widetilde{\boldsymbol{A}}_{22} & \widetilde{\boldsymbol{A}}_{23} \\ \widetilde{\boldsymbol{A}}_{31} & \widetilde{\boldsymbol{A}}_{32} & \widetilde{\boldsymbol{A}}_{33} \end{pmatrix} \text{diag}\,(\bar{\boldsymbol{Q}}_1\ \bar{\boldsymbol{Q}}_2\ \bar{\boldsymbol{Q}}_3)\bar{\Pi}_1$$

$$= \bar{\Pi}_1^T \begin{pmatrix} \tilde{b}_{11}\boldsymbol{I}_2 & \tilde{b}_{12}\boldsymbol{I}_2 & \tilde{b}_{13}\boldsymbol{I}_2 \\ \tilde{b}_{21}\boldsymbol{I}_2 & \tilde{b}_{22}\boldsymbol{I}_2 & \tilde{b}_{23}\boldsymbol{I}_2 \\ \tilde{b}_{31}\boldsymbol{I}_2 & \tilde{b}_{32}\boldsymbol{I}_2 & \tilde{b}_{33}\boldsymbol{I}_2 \end{pmatrix} \bar{\Pi}_1 = \begin{pmatrix} \tilde{\boldsymbol{B}}_1 & \boldsymbol{O} \\ \boldsymbol{O} & \tilde{\boldsymbol{B}}_1 \end{pmatrix},$$

where $\tilde{\boldsymbol{B}}_1 = \begin{pmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \tilde{b}_{13} \\ \tilde{b}_{21} & \tilde{b}_{22} & \tilde{b}_{23} \\ \tilde{b}_{31} & \tilde{b}_{32} & \tilde{b}_{33} \end{pmatrix}.$

Let $\widehat{\boldsymbol{Q}} = \text{diag}\,(\bar{\boldsymbol{Q}}_1\ \bar{\boldsymbol{Q}}_2\ \bar{\boldsymbol{Q}}_3)\bar{\Pi}_1$, which block-diagonalize $\forall \boldsymbol{C} \in \mathcal{T}_1$;

$$\widehat{\boldsymbol{Q}}^T \boldsymbol{C} \widehat{\boldsymbol{Q}} = \begin{pmatrix} \boldsymbol{B}_1 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{B}_1 \end{pmatrix},$$

$\boldsymbol{B}_1 \in$ an irreducible *-algebra $\mathcal{T}_1' = \mathcal{M}_3$.

We assumed $\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{I} + \sum_{p=1}^{m} r_p \boldsymbol{A}_p$ and $\boldsymbol{A}(\boldsymbol{s})$ are generic.

Suppose this assumption is not satisfied.

- The proposed method may fail or the resulting decomposition may not be the finest one.

$\Downarrow$

- A reason may be that $\{t_0 \boldsymbol{I} + \sum_{p=1}^{m} t_p \boldsymbol{A}_p : t_p \in \mathbb{R}\}$ does not span $\mathcal{T} \cap \mathcal{S}_n$.

1. Generate matrices $\boldsymbol{A}_p \boldsymbol{A}_q + \boldsymbol{A}_q \boldsymbol{A}_p \in \mathcal{T} \cap \mathcal{S}_n$ $(1 \leq p \leq q \leq m)$, and add some not included in $\{t_0 \boldsymbol{I} + \sum_{p=1}^{m} t_p \boldsymbol{A}_p : t_p \in \mathbb{R}\}$, say $\boldsymbol{A}_{m+1}, \ldots, \boldsymbol{A}_{m'}$, to $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m$
2. Apply the proposed method to $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_{m'}$.

If we repeat Steps 1 and 2, then $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_{m'}$ eventually span $\mathcal{T} \cap \mathcal{S}_n$ and random choices $\boldsymbol{A}(\boldsymbol{r})$ and $\boldsymbol{A}(\boldsymbol{s})$ from

$$\{t_0 \boldsymbol{I} + \sum_{p=1}^{m'} t_p \boldsymbol{A}_p : t_p \in \mathbb{R}\}$$

is expected to be generic.

Outline

1. Theoretical Framework
2. Numerical Method
3. Some Examples
4. Concluding Remarks

Example 1

$$\boldsymbol{A}_1 = \begin{pmatrix} \boldsymbol{B} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{B} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{B} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \end{pmatrix}, \quad \boldsymbol{A}_2 = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{C} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{C} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{C} \\ \boldsymbol{C} & \boldsymbol{C} & \boldsymbol{C} & \boldsymbol{O} \end{pmatrix},$$

$$\boldsymbol{A}_3 = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{D} \end{pmatrix}, \quad \boldsymbol{A}_4 = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{E} & \boldsymbol{E} & \boldsymbol{O} \\ \boldsymbol{E} & \boldsymbol{O} & \boldsymbol{E} & \boldsymbol{O} \\ \boldsymbol{E} & \boldsymbol{E} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \end{pmatrix}.$$

Case 1: a general case under S_3-symmetry (the symmetric group of degree 3).

$$\boldsymbol{B} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{D} = 1, \boldsymbol{E} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},$$

$$\boldsymbol{P}^T \boldsymbol{A}_p \boldsymbol{P} = \text{diag}\left(\boldsymbol{B}_{p1}, (\boldsymbol{B}_{p2}, \boldsymbol{B}_{p2})\right), \ \boldsymbol{B}_{p1} \in \mathcal{M}_3, \ \boldsymbol{B}_{p2} \in \mathcal{M}_2.$$

Here

$$
\begin{aligned}
\boldsymbol{B}_{11} &= \begin{pmatrix} -0.99648 & -0.07327 & -0.06501 \\ 0.07327 & 0.54451 & 1.15740 \\ -0.06501 & 1.15740 & 2.45197 \end{pmatrix}, \\
\boldsymbol{B}_{21} &= \cdots, \ \boldsymbol{B}_{31} = \cdots, \ \boldsymbol{B}_{41} = \cdots, \\
\boldsymbol{B}_{22} &= \boldsymbol{O}, \boldsymbol{B}_{32} = \boldsymbol{O}, \\
\boldsymbol{B}_{12} &= \begin{pmatrix} -0.99954 & -0.04297 \\ -0.04297 & 2.99954 \end{pmatrix}, \\
\boldsymbol{B}_{42} &= \begin{pmatrix} -1.51097 & 0.52137 \\ 0.52137 & -3.48903 \end{pmatrix}.
\end{aligned}
$$

Example 1

$$A_1 = \begin{pmatrix} B & O & O & O \\ O & B & O & O \\ O & O & B & O \\ O & O & O & O \end{pmatrix}, \; A_2 = \begin{pmatrix} O & O & O & C \\ O & O & O & C \\ O & O & O & C \\ C & C & C & O \end{pmatrix},$$

$$A_3 = \begin{pmatrix} O & O & O & O \\ O & O & O & O \\ O & O & O & O \\ O & O & O & D \end{pmatrix}, \; A_4 = \begin{pmatrix} O & E & E & O \\ E & O & E & O \\ E & E & O & O \\ O & O & O & O \end{pmatrix}.$$

Case 2: a commutativity relation $BE = EB$.

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \; C = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \; D = 1, E = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix},$$

$$P^T A_p P = \text{diag}\,(B_{p1}, (B_{p2}, B_{p2}), (B_{p3}, B_{p3})),$$

$$B_{p1} \in \mathcal{M}_3, \; B_{p2}, \; B_{p3} \in \mathcal{M}_1.$$

Example 1

$$A_1 = \begin{pmatrix} B & O & O & O \\ O & B & O & O \\ O & O & B & O \\ O & O & O & O \end{pmatrix}, \; A_2 = \begin{pmatrix} O & O & O & C \\ O & O & O & C \\ O & O & O & C \\ C & C & C & O \end{pmatrix},$$

$$A_3 = \begin{pmatrix} O & O & O & O \\ O & O & O & O \\ O & O & O & O \\ O & O & O & D \end{pmatrix}, \; A_4 = \begin{pmatrix} O & E & E & O \\ E & O & E & O \\ E & E & O & O \\ O & O & O & O \end{pmatrix}.$$

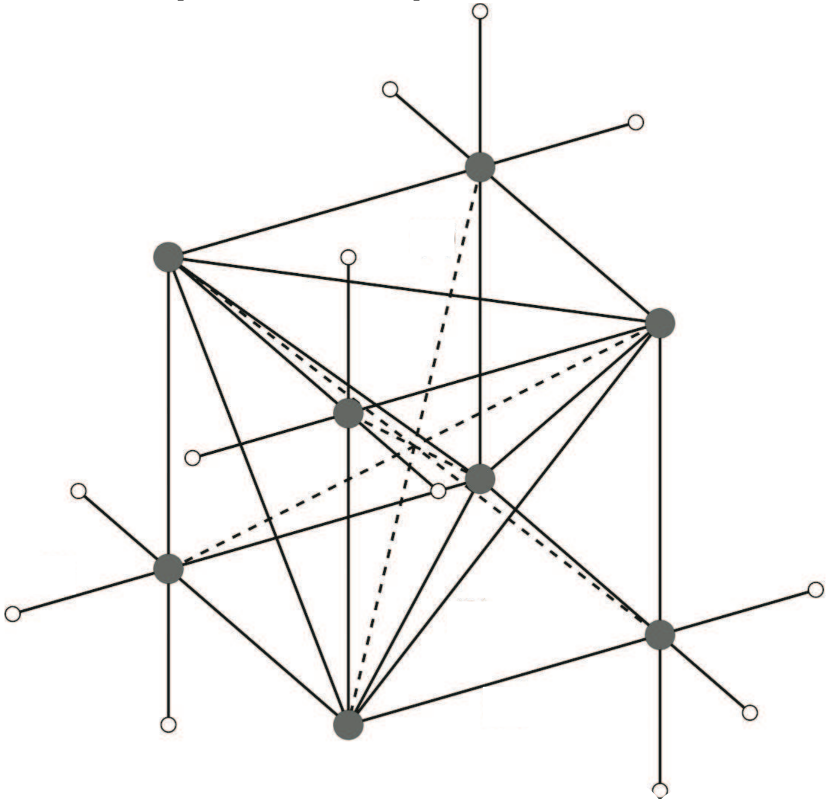
Case 3: Case 2 & C is an eigenvector of B and E.

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \; C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; D = 1, E = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix},$$

$$P^T A_p P = \text{diag}\left(B_{p1}, B_{p4}, (B_{p2}, B_{p2}), (B_{p3}, B_{p3})\right),$$

$$B_{p1} \in \mathcal{M}_2, \; B_{p2}, \; B_{p3}, \; B_{p4} \in \mathcal{M}_1.$$

Example 2: Optimization of cubic trusses



●	:	a free node
—	:	a member of a truss
ℓ_j	:	length of jth member, fixed
η_j	:	cross subsection area of jth member, variable

minimize$_{\eta_j}$ the total volume $\sum_j \ell_j \eta_j$ of the structure

subject to the eigenvalues of vibration $\geq$ a specified value

case 1: 34 members including dotted ones.

case 2: 30 members excluding dotted ones.

Group symetric SDP

$$\begin{array}{ll} \max & -\sum_{p=1}^{m} b_p y_p \\ \text{sub. to} & \boldsymbol{A}_0 - \sum_{p=1}^{m} \boldsymbol{A}_p y_p \succeq \boldsymbol{O}, \\ & y_p \ (p = 1, \ldots, m). \end{array}$$

$\Downarrow$ $\Rightarrow$ the variable-linking tecnique $\Rightarrow$ $\Uparrow$ Here $\boldsymbol{A}_p \in \mathcal{S}_{24}$

Let $\mathcal{T}$ be a *-algebra of $\mathcal{M}_n$ generated by sym. matrices $\boldsymbol{I}, \boldsymbol{A}_0, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_m$. Then $\exists$ orthog. mat. $\boldsymbol{P}$; for $\forall \boldsymbol{A} \in \mathcal{T}$

$$\boldsymbol{P}^T \boldsymbol{A} \boldsymbol{P} = \bigoplus_{j=1}^{\ell} \boldsymbol{C}_j = \bigoplus_{j=1}^{\ell} \bigoplus_{i=1}^{m_j} \boldsymbol{B}_{ji}$$

Here $\boldsymbol{B}_j \in \mathcal{M}_{n_j}$.

	case 1: $\ell = 4$		case 2: $\ell = 5$	
	m_j	n_j	m_j	n_j
$j = 1$	1	2	1	2
$j = 2$	2	2	2	2
$j = 3$	3	2	3	2
$j = 4$	3	4	3	2
$j = 5$	-	-	3	2

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A numerical method for computing an orthogonal matrix $\boldsymbol{P}$ which block-diagoanizes given $n \times n$ symmetric matrices $\boldsymbol{A}_p, \ldots, \boldsymbol{A}_m$.

(a) No information on any algebraic structure is assumed to be know in advance.

(b) Accurate computation of eigenvalues of a symmetric matrix and solutions of linear systems of equations.

- Numerical stability for large size of matrices.

Fundamental theorem behind the proposed method:

Let $\mathcal{T}$ be a *-algebra of $\mathcal{M}_n$ generated by sym. matrices $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m$. Then $\exists$ orthog. mat. $\boldsymbol{P}$;

$$\boldsymbol{P}^T \boldsymbol{A} \boldsymbol{P} = \bigoplus_{j=1}^{\ell} \boldsymbol{C}_j = \bigoplus_{j=1}^{\ell} \text{diag}\,(\boldsymbol{B}_j, \ldots, \boldsymbol{B}_j), \ \forall \boldsymbol{A} \in \mathcal{T}.$$

$\mathcal{T}_j = \left\{ \text{diag}\,(\boldsymbol{B}_j, \ldots, \boldsymbol{B}_j) : \boldsymbol{B}_j \in \mathcal{T}'_j \right\}$: a simple *-algebra.

$\mathcal{T}'_j$ is irreducible, $\mathcal{T}'_j = \mathcal{M}_{n_j}(\mathbb{R})$, $\mathcal{M}_{n_j/2}(\mathbb{C})$ or $\mathcal{M}_{n_j/4}(\mathbb{H})$.

$\mathcal{M}_n(\mathbb{R})$ = the matrix algebra of $n \times n$ real matrices.

$\mathcal{M}_{n/2}(\mathbb{C})$ = "a real representation of the $(n/2 \times n/2)$ complex matrix algebra"

$\mathcal{M}_{n/4}(\mathbb{H})$ = "$\cdots\cdots$ the $(n/4 \times n/4)$ quaternion mat. algebra"

- Some modification to process $\mathcal{M}_{n_j/2}(\mathbb{C})$, $\mathcal{M}_{n_j/4}(\mathbb{H})$ cases.
- Maehara-Murota (June 2008) proposes a numerical method for block-diagonalizing (not necessarily symmetric) $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m \in \mathcal{M}_n$; the method proposed there can process $\mathcal{M}_{n_j/2}(\mathbb{C})$ and $\mathcal{M}_{n_j/4}(\mathbb{H})$ cases.