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Second Order Cone Programming Relaxation of Positive Semidefinite Constraint

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Abstract The positive semidefinite constraint for the variable matrix in semidefinite programming (SDP) relaxation is further relaxed by a finite number of second order cone constraints in second order cone programming (SOCP) relaxations. A few types of SOCP relaxations are obtained from different ways of expressing the positive semidefinite constraint of the SDP relaxation. We present how such SOCP relaxations can be derived, and show the relationship between the resulting SOCP relaxations.

Key words. Convex relaxation, Nonconvex program, Quadratic program, Semidefinite program, Second order cone program

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1 Introduction

It is known that a nonconvex quadratic optimization problem (QOP), a QOP which includes an indefinite quadratic objective function to be minimized and/or indefinite quadratic inequality constraints, is NP-hard. The semidefinite programming (SDP) relaxation is often used to compute effective lower bounds for the minimum objective value of such QOP. A downside of applying the SDP relaxation to a very large size of QOP lies in computational inefficiency to obtain a solution. Solving a large scale SDP problem remains to be a difficult problem though a great deal of efforts continue to be made [2, 3, 4]. As an alternative, we consider second order cone programming (SOCP) relaxation methods for QOPs. Second order cone programming (SOCP) problem is a convex optimization problem to find variable vectors to minimize a linear function subject to linear constraints and constraints that the variable vectors be in the Cartesian product of second order cones [1, 8]. The SOCP relaxation proposed in [5] demonstrated numerical efficiency over the SDP relaxation. In general the bounds obtained by the SOCP relaxation for nonconvex QOPs are inferior to the ones obtained by the SDP relaxation. But the numerical efficiency over the SDP relaxation is a crucial advantage of the SOCP relaxation.

The SOCP relaxation in [5, 6, 9] can be described as relaxing the positive semidefinite constraint for the variable matrix of the SDP relaxation to second order cone constraints. More precisely, from the positive semidefinite constraint for the variable matrix of the SDP relaxation, we select some positive semidefinite conditions that can be relaxed to second order cone constraints. As a result, the SOCP relaxations in [5, 6, 9] represents a transformation from infinite number of linear inequality constraints that the original positive semidefinite constraint implies to a finite number of second order cone constraints, depending on what types of positive semidefinite conditions have been selected.

The SOCP relaxation in the paper [5] by Kim and Kojima relies on the fact that the $n \times n$ symmetric matrix $\boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^T$ is positive semidefinite if and only if its inner product with any $n \times n$ positive semidefinite matrix is nonnegative. They choose a finite number of positive semidefinite matrices from the spectral decomposition of indefinite matrices in the constraints of a given QOP, and require that the inner product of $\boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^T$ with those matrices be nonnegative. The resulting constraints form a finite number of convex quadratic inequalities, which are transformed to the same number of second order cone constraints. We call this the SOCP relaxation of type 1.

A different SOCP relaxation is proposed in their paper [6] by focusing on the variable matrix of an SDP problem. They used two facts to derive a finite number of second order cone constraints. The one is that every 2×2 principle submatrix of a positive semidefinite matrix is also a positive semidefinite matrix. The other is that the positive semidefiniteness of each 2×2 symmetric matrix is represented as a second order cone constraint. Thus the positive semidefinite condition imposed on the $n \times n$ variable matrix of the SDP problem is relaxed into at most $n(n-1)/2$ second order cone constraints. This is called as the SOCP relaxation of type 2 in this paper.

From the quality of bounds for objective values of QOPs, the SDP relaxation is more effective than the SOCP relaxation in general, though both relaxations provide the same optimal value for some class of QOPs [6]. The SOCP relaxation is, however, stronger than lift-and-project linear programming relaxation by this construction. As far as numerical efficiency is concerned, the SOCP relaxations of type 1 and 2 showed much better performance than the SDP relaxation as in numerical results in [5, 6]. The effectiveness of an SOCP relaxation is determined by how well the positive semidefinite constraint for the variable matrix in the SDP relaxation is represented. We note that different representations of the positive semidefinite constraint in the SDP relaxation leads to various SOCP relaxations, and their effectiveness and efficiency also vary.

The main purpose of this paper is to present an extension of the SOCP relaxation of type 1, which provides us with flexibility in formulating SOCP relaxations, and to derive its characterization in terms of a family of linear inequalities.

This paper is organized as follows: in Section 2, we describe a QOP and its SDP relaxation. Section 3 contains how to obtain SOCP relaxations from the positive semidefinite constraint of the SDP relaxation. Specifically, we present an extension of the SOCP relaxation of type 1 and its characterization. In Section 4, the relationship between the SOCP relaxation of type 1 and the SOCP relaxation of type 2 is described. We give some concluding discussions in Section 5.

Throughout the paper, we use the following notation: Let $\mathbb{R}^n$, $\mathcal{S}^n$ and $\mathcal{S}^n_+$ denote the n-dimensional Euclidean space, the set of $n \times n$ symmetric matrices and the set of $n \times n$ positive semidefinite symmetric matrices, respectively. For $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{S}^n$, we denote $\boldsymbol{A} \bullet \boldsymbol{B} = \sum_{i=1}^n \sum_{j=1}^n \boldsymbol{A}_{ij} \boldsymbol{B}_{ij}$, and $\boldsymbol{A} \succeq \boldsymbol{B}$ means $\boldsymbol{A} - \boldsymbol{B}$ is positive semidefinite.

2 A quadratic optimization problem and its SDP relaxation

Consider a QOP of the form

$$\left.\begin{array}{ll} \text{minimize} & \boldsymbol{x}^T \boldsymbol{Q}_0 \boldsymbol{x} + \boldsymbol{q}_0^T \boldsymbol{x} \\ \text{subject to} & \boldsymbol{x}^T \boldsymbol{Q}_p \boldsymbol{x} + \boldsymbol{q}_p^T \boldsymbol{x} + \gamma_p \leq 0 \quad (1 \leq p \leq m). \end{array}\right\} \tag{1}$$

Here $\boldsymbol{Q}_p$ is an $n \times n$ symmetric matrix, $\boldsymbol{q}_p \in \mathbb{R}^n$, and $\gamma_p \in \mathbb{R}$ for $0 \leq p \leq m$.

An SDP relaxation of the QOP (1) is

$$\left.\begin{array}{ll} \text{minimize} & \boldsymbol{Q}_0 \bullet \boldsymbol{X} + \boldsymbol{q}_0^T \boldsymbol{x} \\ \text{subject to} & \boldsymbol{Q}_p \bullet \boldsymbol{X} + \boldsymbol{q}_p^T \boldsymbol{x} + \gamma_p \leq 0 \quad (1 \leq p \leq m), \\ & \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{X} \end{pmatrix} \in \mathcal{S}^{1+n}_+. \end{array}\right\} \tag{2}$$

It is the the positive semidefinite condition

$$\boldsymbol{Y} = \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{X} \end{pmatrix} \in \mathcal{S}^{1+n}_+ \tag{3}$$

of the SDP relaxation (2) that we focus our attention in the remainder of this paper. We consider the SOCP relaxations of types 1 and 2 for (3).

3 An extension of the SOCP relaxation of type 1

We first describe the SOCP relaxation of type 1 briefly. The positive semidefinite condition (2) is equivalent to the positive semidefinite condition $\boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^T \in \mathcal{S}^n_+$ on the smaller matrix $\boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^T$. The latter condition holds if and only if $\boldsymbol{C} \bullet \left(\boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^T\right) \geq 0$ is true for every $\boldsymbol{C} \in \mathcal{S}^n_+$. This inequality, in turn, can be written as

$$\boldsymbol{x}^T\boldsymbol{C}\boldsymbol{x} - \boldsymbol{C} \bullet \boldsymbol{X} \leq 0.$$

Note that the left hand side of the inequality with a fixed $\boldsymbol{C} \in \mathcal{S}^n_+$ is convex in $\boldsymbol{x}$ and linear in $\boldsymbol{X}$; hence we can rewrite the inequality as an SOCP constraint [5].

We extend the SOCP relaxation of type 1 by considering a more general partition of the variable matrix $\boldsymbol{Y} \in \mathcal{S}^{1+n}$ Now, let

$$\boldsymbol{Y} \;=\; \begin{pmatrix} \boldsymbol{Z} & \boldsymbol{z} & \boldsymbol{V} \\ \boldsymbol{z}^T & \zeta & \boldsymbol{w}^T \\ \boldsymbol{V}^T & \boldsymbol{w} & \boldsymbol{W} \end{pmatrix} \in \mathcal{S}^{k+1+(n-k)}.$$

First we derive some necessary conditions for $\boldsymbol{Y}$ to be positive semidefinite. For simplicity, we assume that $k = n$ or

$$\boldsymbol{Y} = \begin{pmatrix} \boldsymbol{Z} & \boldsymbol{z} \\ \boldsymbol{z}^T & \zeta \end{pmatrix} \in \mathcal{S}^{n+1}. \tag{4}$$

But all the discussions and results remain valid with slight modifications.

Lemma 3.1. *$\boldsymbol{Y} \succeq \boldsymbol{O}$ if and only if $\zeta \geq 0$, $\boldsymbol{Z} \succeq \boldsymbol{O}$ and $\zeta\boldsymbol{Z} - \boldsymbol{z}\boldsymbol{z}^T \succeq \boldsymbol{O}$.*

Proof: First assume that $\zeta = 0$. Then $\boldsymbol{Y} \succeq \boldsymbol{O}$ if and only if $\boldsymbol{Z} \succeq \boldsymbol{O}$ and $\boldsymbol{z} = \boldsymbol{0}$. Since $\boldsymbol{z} = \boldsymbol{0}$ if and only if $-\boldsymbol{z}\boldsymbol{z}^T \succeq \boldsymbol{O}$, the desired result follows. Now assume that $\zeta > 0$. We observe that

$$\begin{pmatrix} \zeta\boldsymbol{Z} - \boldsymbol{z}\boldsymbol{z}^T & \boldsymbol{0} \\ \boldsymbol{0}^T & \zeta \end{pmatrix} \;=\; \begin{pmatrix} \sqrt{\zeta}\boldsymbol{I} & -\boldsymbol{z}/\sqrt{\zeta} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{Z} & \boldsymbol{z} \\ \boldsymbol{z}^T & \zeta \end{pmatrix} \begin{pmatrix} \sqrt{\zeta}\boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{z}^T/\sqrt{\zeta} & 1 \end{pmatrix}.$$

Since the matrix $\begin{pmatrix} \sqrt{\zeta}\boldsymbol{I} & -\boldsymbol{z}/\sqrt{\zeta} \\ \boldsymbol{0}^T & 1 \end{pmatrix}$ is nonsingular in this case, the desired result follows. ∎

Let $\boldsymbol{A}$ be an $m \times m$ positive definite matrix, $\boldsymbol{B}$ an $n \times m$ matrix and $\boldsymbol{Z}$ an $n \times n$ symmetric matrix. Then the following relation is well-known (for example, see Theorem 6.13 in page 178 [10]):

$$\boldsymbol{Y} = \begin{pmatrix} \boldsymbol{Z} & \boldsymbol{B} \\ \boldsymbol{B}^T & \boldsymbol{A} \end{pmatrix} \succeq \boldsymbol{0} \quad\Leftrightarrow\quad \boldsymbol{Z} \succeq \boldsymbol{0} \text{ and } \boldsymbol{Z} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}^T \succeq \boldsymbol{0}.$$

The matrix $\boldsymbol{Z} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}^T$ is often called the Schur complement of $\boldsymbol{Y}$. A difference between Lemma 3.1 and the relation above is that Lemma 3.1 includes the case $\zeta = 0$; if we restrict ζ to be positive, it is obvious that Lemma 3.1 is a special case of the above relation.

As a corollary of Lemma 3.1, we obtain:

Lemma 3.2. *Suppose that $\boldsymbol{Y}$ is an $(n+1)\times(n+1)$ matrix of the form (4). Then $\boldsymbol{Y} \succeq \boldsymbol{O}$ if and only if*

$$\zeta \geq 0,\ \boldsymbol{C} \bullet \boldsymbol{Z} \geq 0 \ \textit{ and } \boldsymbol{C} \bullet \left(\zeta \boldsymbol{Z} - \boldsymbol{z}\boldsymbol{z}^T\right) \geq 0 \tag{5}$$

for every $\boldsymbol{C} \succeq \boldsymbol{O}$.

Let $\mathcal{S}^n \ni \boldsymbol{C} \succeq \boldsymbol{O}$ be fixed. Suppose that $q = \text{rank } C \geq 1$. Then we can take a $q \times n$ matrix $\boldsymbol{U}$ such that $\boldsymbol{C} = \boldsymbol{U}^T\boldsymbol{U}$. Hence we can rewrite (5) in Lemma 3.2 above as

$$\zeta \geq 0,\ \boldsymbol{C} \bullet \boldsymbol{Z} \geq 0 \ \text{ and } (\boldsymbol{U}\boldsymbol{z})^T(\boldsymbol{U}\boldsymbol{z}) \leq \zeta(\boldsymbol{C} \bullet \boldsymbol{Z}).$$

It is well-known that these conditions can be transformed into a second order cone inequality

$$\left\| \begin{pmatrix} \zeta - \boldsymbol{C} \bullet \boldsymbol{Z} \\ 2\,\boldsymbol{U}\boldsymbol{z} \end{pmatrix} \right\| \leq \zeta + \boldsymbol{C} \bullet \boldsymbol{Z}. \tag{6}$$

Lemma 3.2 shows that a positive semidefinite constraint $\boldsymbol{Y} \succeq \boldsymbol{0}$ is equivalent to infinite number of second order cone constraints since $\boldsymbol{C}$ can be chosen as any element of the set of positive semidefinite matrices. As in the SOCP relaxation of type 1 [5], we choose a finite number positive semidefinite matrices, $\boldsymbol{C}_1, \boldsymbol{C}_2, \ldots, \boldsymbol{C}_\ell \succeq \boldsymbol{0}$. As a result, we have an SOCP relaxation of the positive semidefinite constraint $\boldsymbol{Y} \succeq \boldsymbol{O}$ as follows:

$$\left\| \begin{pmatrix} \zeta - \boldsymbol{C}_i \bullet \boldsymbol{Z} \\ 2\,\boldsymbol{U}_i\boldsymbol{z} \end{pmatrix} \right\| \leq \zeta + \boldsymbol{C}_i \bullet \boldsymbol{Z} \quad (i = 1, 2, \ldots, \ell), \tag{7}$$

where $\boldsymbol{C}_i = {\boldsymbol{U}_i}^T\boldsymbol{U}_i$.

Now we mention our main result, which characterizes the necessary condition (5) for $\boldsymbol{Y}$ of the form (4) to be positive semidefinite or the SOCP constraint (6) in terms of a family of linear inequalities.

Theorem 3.3. *Let $\mathcal{S}^n \ni \boldsymbol{C} \succeq \boldsymbol{O}$ be fixed. Then (5) (hence (6)) holds if and only if*

$$\boldsymbol{B} \bullet \begin{pmatrix} \boldsymbol{Z} & \boldsymbol{z} \\ \boldsymbol{z}^T & \zeta \end{pmatrix} \geq 0 \tag{8}$$

for every $\boldsymbol{B} \succeq \boldsymbol{O}$ of the form $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{b} \\ \boldsymbol{b}^T & \gamma \end{pmatrix}$.

Proof: (i) Only if part: Assume that (5) holds. Let $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{b} \\ \boldsymbol{b}^T & \gamma \end{pmatrix} \succeq \boldsymbol{O}$. Then we see that

$$0 \ \leq \ (\boldsymbol{z}^T,\ \zeta) \begin{pmatrix} \boldsymbol{C} & \boldsymbol{b} \\ \boldsymbol{b}^T & \gamma \end{pmatrix} \begin{pmatrix} \boldsymbol{z} \\ \zeta \end{pmatrix} = \boldsymbol{z}^T\boldsymbol{C}\boldsymbol{z} + 2\zeta\boldsymbol{b}^T\boldsymbol{z} + \gamma\zeta^2.$$

Hence, taking the assumption (5) into account, we obtain that

$$0 \leq \zeta \boldsymbol{C} \bullet \boldsymbol{Z} - \boldsymbol{z}^T\boldsymbol{C}\boldsymbol{z} \leq \zeta \boldsymbol{C} \bullet \boldsymbol{Z} + 2\zeta\boldsymbol{b}^T\boldsymbol{z} + \gamma\zeta^2. = \zeta \boldsymbol{B} \bullet \begin{pmatrix} \boldsymbol{Z} & \boldsymbol{z} \\ \boldsymbol{z}^T & \zeta \end{pmatrix}.$$

Hence if $\zeta > 0$ then $\boldsymbol{B} \bullet \begin{pmatrix} \boldsymbol{Z} & \boldsymbol{z} \\ \boldsymbol{z}^T & \zeta \end{pmatrix} \geq 0$. Now assume that $\zeta = 0$. In this case. the assumption (5) implies that $\boldsymbol{z}^T\boldsymbol{C}\boldsymbol{z} = 0$. In addition, we know that for any $\delta \in \mathbb{R}$

$$0 \;\leq\; (\boldsymbol{z}^T, \delta) \begin{pmatrix} \boldsymbol{C} & \boldsymbol{b} \\ \boldsymbol{b}^T & \gamma \end{pmatrix} \begin{pmatrix} \boldsymbol{z} \\ \delta \end{pmatrix} = \gamma\delta^2 + 2\delta \boldsymbol{b}^T\boldsymbol{z}.$$

Hence $\boldsymbol{b}^T\boldsymbol{z} = 0$. Therefore we obtain

$$\begin{pmatrix} \boldsymbol{C} & \boldsymbol{b} \\ \boldsymbol{b}^T & \gamma \end{pmatrix} \bullet \begin{pmatrix} \boldsymbol{Z} & \boldsymbol{z} \\ \boldsymbol{z}^T & \zeta \end{pmatrix} = \boldsymbol{C} \bullet \boldsymbol{Z} \geq 0.$$

Thus we have shown the "only if" part.

(ii) If part: Assume that (8) holds for every $\boldsymbol{B} \succeq \boldsymbol{O}$ of the form $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{b} \\ \boldsymbol{b}^T & \gamma \end{pmatrix}$. Specifically for the case $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0}^T & \gamma \end{pmatrix}$,

$$\boldsymbol{B} \bullet \begin{pmatrix} \boldsymbol{Z} & \boldsymbol{z} \\ \boldsymbol{z}^T & \zeta \end{pmatrix} = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0}^T & \gamma \end{pmatrix} \bullet \begin{pmatrix} \boldsymbol{Z} & \boldsymbol{z} \\ \boldsymbol{z}^T & \zeta \end{pmatrix} \geq 0 \text{ for every } \gamma \geq 0.$$

This implies that $\zeta \geq 0$ and $\boldsymbol{C} \bullet \boldsymbol{Z} \geq 0$. For every $\epsilon > 0$, let us consider the identies

$$\begin{aligned} \boldsymbol{B}(\epsilon) &= \begin{pmatrix} \boldsymbol{C} & -\boldsymbol{C}\boldsymbol{z}/(\epsilon+\zeta) \\ -\boldsymbol{z}^T\boldsymbol{C}/(\epsilon+\zeta) & \boldsymbol{z}^T\boldsymbol{C}\boldsymbol{z}/(\epsilon+\zeta)^2 \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{z}^T/(\epsilon+\zeta) & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0}^T & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{z}/(\epsilon+\zeta) \\ \boldsymbol{0}^T & 1 \end{pmatrix}. \end{aligned}$$

Then $\boldsymbol{B}(\epsilon)$ is a positive semidefinite matrix of the form under consideration. Hence

$$\begin{aligned} 0 &\leq \boldsymbol{B}(\epsilon) \bullet \begin{pmatrix} \boldsymbol{Z} & \boldsymbol{z} \\ \boldsymbol{z} & \epsilon+\zeta \end{pmatrix} \\ &= \boldsymbol{z}^T\boldsymbol{C}\boldsymbol{z}/(\epsilon+\zeta) - 2\boldsymbol{z}^T\boldsymbol{C}\boldsymbol{z}/(\epsilon+\zeta) + \boldsymbol{C} \bullet \boldsymbol{Z} \\ &= \left((\epsilon+\zeta)\boldsymbol{C} \bullet \boldsymbol{Z} - \boldsymbol{z}^T\boldsymbol{C}\boldsymbol{z}\right)/(\epsilon+\zeta). \end{aligned}$$

Hence $0 \leq (\epsilon+\zeta)\boldsymbol{C} \bullet \boldsymbol{Z} - \boldsymbol{z}^T\boldsymbol{C}\boldsymbol{z}$. Taking the limit as $\epsilon \to 0$, we consequently obtain that $0 \leq \zeta\boldsymbol{C} \bullet \boldsymbol{Z} - \boldsymbol{z}^T\boldsymbol{C}\boldsymbol{z}$. Thus we have shown the "if part". ∎

From Theorem 3.3, we see that for every $\boldsymbol{B} \succeq \boldsymbol{O}$, (8) can be turned to (6). Notice that in (6), we only have the positive semidefinite matrix $\boldsymbol{C}$; the vector $\boldsymbol{b}$ and the number γ do not appear. This provides a reduced second order cone representation of a family of linear inequalities (8) with $\boldsymbol{B} \succeq \boldsymbol{O}$ of the form $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{b} \\ \boldsymbol{b}^T & \gamma \end{pmatrix}$.

4 The SOCP relaxation of type 2

We show the SOCP relaxation of type 2 can be viewed as a special case of the SOCP relaxation of type 1.

If $\boldsymbol{Y}$ of the form (4) is positive semidefinite, then so is each 2×2 principle matrix. Then, we can rewrite this as an SOCP inequality. For simplicity, we consider the case

$$\begin{pmatrix} Z_{nn} & z_n \\ z_n & \zeta \end{pmatrix} \succeq \boldsymbol{O}. \tag{9}$$

We know that this relation holds if and only if

$$\zeta \geq 0,\ Z_{nn} \geq 0 \ \text{ and } \zeta Z_{nn} \geq z_n^2.$$

Furthermore we can convert the set of inequalities above into an SOCP inequality

$$\left\| \begin{pmatrix} \zeta - Z_{nn} \\ 2z_n \end{pmatrix} \right\| \leq \zeta + Z_{nn}. \tag{10}$$

Now, we consider the relationship of the SOCP relaxations of type 1 and 2. Let

$$\boldsymbol{U} = \begin{pmatrix} \boldsymbol{0}^T & 1 \end{pmatrix} \in \mathbb{R}^n \quad \text{and} \quad \boldsymbol{C} = \boldsymbol{U}^T \boldsymbol{U} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix}.$$

Then, we can see that the SOCP inequality (6) coincides with the SOCP inequality (10). Thus we may regard the SOCP relaxation in this section as a special case of the SOCP relaxation described in Section 3.

5 Concluding Remarks

We have shown how the positive semidefinite constraint of the SDP relaxation can be relaxed to second order cone constraints in the SOCP relaxations. Different ways of choosing positive semidefinite coefficient matrices yield different SOCP relaxations. We have extended the SOCP relaxation in [5] by considering a more general form of the variable matrix of the SDP relaxation. This gives us flexibility to formulate SOCP relaxations, especially when deriving an effective SOCP relaxation is an important issue.

In recent work by Kojima *et al* [7], a new framework for convex relaxation of polynomial optimization problems over cones in terms of linear optimization problems (LOPs) over cones was presented. The framework provided various ways of formulating convex relaxation using LOPs over cones, of which SOCP relaxation was shown as a special case. Since obtaining a good bound for a general QOP by applying the SDP relaxation once is hard to expect and achieve, repeated use of convex relaxation methods is necessary, and therefore effective and efficient convex relaxation methods are in demand. In this perspective, providing different ways to derive SOCP relaxations can be regarded as an advantage of choosing SOCP relaxations among many convex relaxation methods.

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