Computing All Nonsingular Solutions of Cyclic-n Polynomial Using Polyhedral Homotopy Continuation Methods

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Abstract. All isolated solutions of the cyclic-n polynomial equations are not known for larger dimensions than 11. We exploit two types of symmetric structures in the cyclic-n polynomial to compute all isolated nonsingular solutions of the equations efficiently by the polyhedral homotopy continuation method and to verify the correctness of the generated approximate solutions. Numerical results on the cyclic-8 to the cyclic-12 polynomial equations, including their solution information, are given.

Key words. Polynomial, Cyclic Polynomial, Homotopy Continuation Methods, Polyhedral Homotopy, Cheater's Homotopy.

1 Introduction

We consider solving a system of the *cyclic-n polynomial* equations [3]

$$\boldsymbol{f}^c(\boldsymbol{x}) = (f_1^c(\boldsymbol{x}), f_2^c(\boldsymbol{x}), \ldots, f_n^c(\boldsymbol{x})) = \mathbf{0}$$

with homotopy continuation methods. Here each component $f_j^c(\boldsymbol{x})$ is defined as

$$\begin{array}{rcl} f_1^c(\boldsymbol{x}) & = & x_1 + x_2 + \cdots + x_n, \\ f_2^c(\boldsymbol{x}) & = & x_1x_2 + x_2x_3 + \cdots + x_nx_1, \\ & & \ldots \\ f_{n-2}^c(\boldsymbol{x}) & = & x_1x_2 \ldots x_{n-2} + x_2x_3 \ldots x_{n-1} + \cdots + x_nx_1 \cdots x_{n-2}, \\ f_{n-1}^c(\boldsymbol{x}) & = & x_1x_2 \ldots x_{n-1} + x_2x_3 \ldots x_n + \cdots + x_nx_1 \cdots x_{n-1}, \\ f_n^c(\boldsymbol{x}) & = & x_1x_2 \ldots x_{n-1}x_n - 1 \end{array}$$

for every $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ and $\mathbb{C}^n$ denotes the n-dimensional complex space. This problem is known as one of the most difficult and challenging bench mark problems [19] for testing numerical methods to find all isolated solutions of polynomial systems.

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Given a system of n polynomials $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x}))$, the basic approach of homotopy continuation methods for solving $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ is to define a system of *homotopy* equations

$$\boldsymbol{h}(\boldsymbol{x}, t) = (h_1(\boldsymbol{x}, t), h_2(\boldsymbol{x}, t), \ldots, h_n(\boldsymbol{x}, t)) = \boldsymbol{0}$$

with a continuation parameter $t \in [0, 1]$ using the algebraic structure of the polynomial system. The homotopy system is constructed so that all solutions of the start system $\boldsymbol{h}(\boldsymbol{x}, 0) = \boldsymbol{0}$ are easily computed and that the target system $\boldsymbol{h}(\boldsymbol{x}, 1) = \boldsymbol{0}$ coincides with the system $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ to be solved. Then we trace solution curves of the homotopy system from $t = 0$ until $t = 1$ to compute solutions of $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{x}, 1) = \boldsymbol{0}$ by predictor and corrector procedures.

The number of homotopy curves that link the start to the target systems determines the number of solutions of $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ that we can compute. Hence, in order to attain all isolated solutions of $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$, each solution of the target system $\boldsymbol{h}(\boldsymbol{x}, 1) = \boldsymbol{0}$ needs to be connected to a solution of the start system $\boldsymbol{h}(\boldsymbol{x}, 0) = \boldsymbol{0}$ through a homotopy curve, while some solutions of the start system may reach none of the solutions of the target system but diverge as $t \to 1$. The polyhedral homotopy based on Bernshtein theory [2, 8, 13, 20], which bounds the number of the isolated zeros of $\boldsymbol{f}(\boldsymbol{x})$ by the mixed volume, provides much fewer homotopy curves to follow than the classical linear homotopy continuation method [1, 6, 11]. The mixed volume is known to give a tighter bound than Bézout bound for the number of solutions in $(\mathbb{C}\backslash\{\boldsymbol{0}\})^n$. When the coefficients of $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ are chosen randomly, the mixed volume or the BKK bound is exact generically.

The polyhedral homotopy functions are constructed on the mixed cells of a polynomial system whose total volume amounts to the mixed volume. Each mixed cell induces a polyhedral homotopy function, which then gives some of the homotopy curves to be traced. Thus all isolated solutions of the polynomial system are obtained by tracing the homotopy curves originated from all mixed cells. For the computation of the mixed cells, several lifting methods such as static, dynamic, and symmetric lifting [21, 22] can be utilized depending on the characteristics of polynomial systems. Static lifting is a general procedure which we can apply to any polynomial system. Symmetric lifting exploits permutation symmetries in a polynomial system, and generates families of mixed cells such that each family is symmetric to a number of other families. Families that are symmetric to each other produce symmetric collections of homotopy curves. Since we only need to trace collections of homotopy curves with different symmetric structure, the number of homotopy curves to be followed is reduced.

Numerical methods for solving polynomial systems using homotopy continuation can be categorized into two groups. One is the methods based on the application of Bézout theorem to count the solutions. Publically available software are CONSOL [15] and HOMPACK [23, 24]. Many extraneous curves must be traced in these methods, affecting numerical efficiency critically for large dimensional problems. As a consequence, the sizes of the cyclic polynomial problems that can be solved by the approach are more restricted than the methods using polyhedral homotopy continuation. PHCpack [22] is one of the most successful polynomial system solvers with polyhedral homotopy continuation, representing the other group of the numerical methods. Currently available solution information of the cyclic-n polynomial problems has been obtained using PHCpack. The problems have been solved employing one type of the symmetric structures, and applying the symmetric lifting and polyhedral continuation methods for the dimension $n = 5\text{-}8, 10, 11$ with PHCpack [19].

Using Groebner bases, Faugère [5] computed some information about the components and the number of isolated solutions of the cyclic-9 polynomial problem. See also [17]. However, the sizes of the cyclic-n polynomial problems that have been solved successfully are still very limited; the solutions of the cyclic polynomial problems with a dimension larger than 11 are not known. That is because in part the problems are often not well-conditioned and the number of homotopy curves to be traced is too large to handle with a single computer.

The aim of this paper is to solve the cyclic-n polynomial problems with polyhedral homotopy methods by exploiting two types of symmetric structures. The symmetric structures are used to decrease the number of homotopy curves and check the correctness of the solutions. Two most important factors in computing all solutions successfully are the number of homotopy curves and tools to validate numerical results at the end of homotopy continuation procedure. The number of homotopy curves is decided by lifting methods and the mixed volume. The mixed volume increases immensely with growing dimensions. Pursuing for the solutions of the dimension 9 and higher dimensional cases of the cyclic polynomial problems involves difficult issues of evaluating all solutions numerically and checking the correctness of numerical results. The strategy here is to use a different type of symmetry in the cyclic-n problems (type-2 symmetry) from the one used in symmetric lifting (type-1 symmetry). We use static lifting and take advantage of the type-2 symmetric structure to reduce the number of homotopy curves to be traced. The structure of symmetry type-2 also plays an important role when examining the correctness of the solutions obtained at $t = 1$.

This paper is organized as follows: Section 2 contains the description of the two types of symmetric structures of the cyclic-n polynomial equations. In Section 3, we address numerical aspects of homotopy continuation methods. Section 4 includes applications of the homotopy continuation methods to the cyclic-n problems. In Section 5, implementation issues in the polyhedral homotopy continuation are discussed. We present numerical results on the cyclic-n polynomial equations with the dimensions $n = 8$ through $n = 12$ in Section 6. Section 7 is devoted to concluding remarks.

We introduce notation and symbols for our succeeding discussions. Let $\mathbb{R}$ and $\mathbb{Z}_+$ denote the set of real numbers and the set of nonnegative integers, respectively. For every variable vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ and every $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_+$, we use the notation $\boldsymbol{x}^{\boldsymbol{a}}$ for the term $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. Then we can write any polynomial $\phi(\boldsymbol{x})$ in the variable vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ as $\phi(\boldsymbol{x}) = \sum_{\boldsymbol{a} \in \mathcal{A}} c(\boldsymbol{a}) \boldsymbol{x}^a$ for some finite subset $\mathcal{A}$ of $\mathbb{Z}^n_+$ and some $c(\boldsymbol{a}) \in \mathbb{C}$ ($\boldsymbol{a} \in \mathcal{A}$). We call $\mathcal{A}$ the *support* of the polynomial $\phi(\boldsymbol{x})$, $\sum_{j=1}^n a_j$ the degree of a term $c(\boldsymbol{a})\boldsymbol{x}^{\boldsymbol{a}}$, and $\max_{\boldsymbol{a} \in \mathcal{A}} \sum_{j=1}^n a_j$ the degree of the polynomial $\phi(\boldsymbol{x})$.

2 Symmetric structures in the solution set of the cyclic-n polynomial equation

Let $\mathcal{A}^c_j$ denote the support of the jth component of $\boldsymbol{f}^c(\boldsymbol{x})$ ($j = 1, 2, \ldots, n$);

$$\left.\begin{array}{ll}
\mathcal{A}^c_1 & = \{\boldsymbol{e}^1, \boldsymbol{e}^2, \ldots, \boldsymbol{e}^n\}, \\
\mathcal{A}^c_2 & = \{\boldsymbol{e}^1 + \boldsymbol{e}^2, \boldsymbol{e}^2 + \boldsymbol{e}^3, \ldots, \boldsymbol{e}^n + \boldsymbol{e}^1\}, \\
 & \quad \cdots \\
\mathcal{A}^c_{n-1} & = \{\boldsymbol{e}^1 + \boldsymbol{e}^2 + \cdots + \boldsymbol{e}^{n-1}, \boldsymbol{e}^2 + \boldsymbol{e}^3 + \cdots + \boldsymbol{e}^n, \ldots, \boldsymbol{e}^n + \boldsymbol{e}^1 + \cdots + \boldsymbol{e}^{n-2}\}, \\
\mathcal{A}^c_n & = \{\boldsymbol{e}, \boldsymbol{0}\}.
\end{array}\right\} \quad (1)$$

Here $\boldsymbol{e}^k \in \mathbb{R}^n$ denotes the kth coordinate vector with 1 in the kth element and 0 elsewhere $(k = 1, 2, \ldots, n)$, $\boldsymbol{e} \in \mathbb{R}^n$ the vector of 1's, and $\boldsymbol{0} \in \mathbb{R}^n$ the vector of 0's, respectively. Then we can rewrite the cyclic-n polynomial as

$$f_j^c(\boldsymbol{x}) \;=\; \sum_{\boldsymbol{a} \in \mathcal{A}_j^c} c_j^c(\boldsymbol{a}) \boldsymbol{x}^{\boldsymbol{a}} \; (j = 1, 2, \ldots, n)$$

for some $c_j^c(\boldsymbol{a}) \in \mathbb{C}$ $(\boldsymbol{a} \in \mathcal{A}_j^c,\ j = 1, 2, \ldots, n)$; in fact,

$$c_j^c(\boldsymbol{a}) = 1 \ (\boldsymbol{a} \in \mathcal{A}_j^c,\ j = 1, 2, \ldots, n-1),\ c_n^c(\boldsymbol{e}) = 1 \ \text{ and } c_n^c(\boldsymbol{0}) = -1. \tag{2}$$

We call a solution $\boldsymbol{x}$ of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$ *nonsingular* if the Jacobian matrix $\boldsymbol{Df}(\boldsymbol{x})$ is nonsingular, and *singular* otherwise. It has been learned that the cyclic-8, -9 and -12 polynomial equations have singular solutions. However, whether the cyclic-n polynomial equations for a general n have singular solutions is still an open problem. Let Σ and $\widetilde{\Sigma} \subseteq \Sigma$ denote the set of all solutions and the set of all nonsingular solutions of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$, respectively. Notice that $x_j \neq 0$ $(j = 1, 2, \ldots, n)$ for any $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Sigma$ since $f_n^c(\boldsymbol{x}) = 0$ implies that none of x_j $(j = 1, 2, \ldots, n)$ can be zero.

2.1 Type-1 symmetric structure

Let $\boldsymbol{P}_f$ and $\boldsymbol{P}_r$ be permutation matrices such that $\boldsymbol{P}_f = (\boldsymbol{e}^2, \boldsymbol{e}^3, \ldots, \boldsymbol{e}^n, \boldsymbol{e}^1)$ and $\boldsymbol{P}_r = (\boldsymbol{e}^n, \boldsymbol{e}^{n-1}, \ldots, \boldsymbol{e}^2, \boldsymbol{e}^1)$. Then $\boldsymbol{x} \in \Sigma$ (*i.e.*, $\boldsymbol{x}$ is a solution of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$) iff $\boldsymbol{P}_f^j \boldsymbol{P}_r^k \boldsymbol{x} \in \Sigma$ for any $j \in \{0, 1, \ldots, n-1\}$ and any $k \in \{0, 1\}$. This symmetric structure of the solution set Σ of the cyclic-n polynomial equations $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$, which we call the type-1 symmetric structure, is well-known [21]. Note that one solution $\boldsymbol{x} \in \Sigma$ is expanded to $2n$ solutions $\boldsymbol{x},\ \boldsymbol{P}_f\boldsymbol{x},\ \boldsymbol{P}_f^2\boldsymbol{x},\ \ldots, \boldsymbol{P}_f^{n-1}\boldsymbol{x},\ \boldsymbol{P}_r\boldsymbol{x},\ \boldsymbol{P}_f\boldsymbol{P}_r\boldsymbol{x},\ \boldsymbol{P}_f^2\boldsymbol{P}_r\boldsymbol{x},\ \ldots, \boldsymbol{P}_f^{n-1}\boldsymbol{P}_r\boldsymbol{x}$ in Σ. Moreover, if $\boldsymbol{x} \in \widetilde{\Sigma}$ (*i.e.*, $\boldsymbol{x}$ is a nonsingular solution of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$)), such expanded solutions form an *orbit of type-1* of $2n$ solutions of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$. Since the cardinality of the set $\widetilde{\Sigma}$ of nonsingular solutions of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$ is finite, $\widetilde{\Sigma}$ can be partitioned into a finite number, *e.g.* $m(n)$, of orbits of type-1, say $\widetilde{\Sigma}_1^1, \widetilde{\Sigma}_2^1, \ldots, \widetilde{\Sigma}_{m(n)}^1$. Here we assume that the cardinality of $\widetilde{\Sigma}$ is $2nm(n)$.

We observe symmetries in both of the supports $\mathcal{A}_j^c$ $(j = 1, 2, \ldots, n)$ (see (1)) and the coefficients $c_j^c(\boldsymbol{a})$ $(\boldsymbol{a} \in \mathcal{A}_j^c,\ j = 1, 2, \ldots, n)$ (see (2)) of $\boldsymbol{f}^c(\boldsymbol{x})$. The type-1 symmetric structure in the solution set $\widetilde{\Sigma}$ of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$ is induced from both of them. Especially, if we modify some of the coefficients of $\boldsymbol{f}^c(\boldsymbol{x})$, the type-1 symmetric structure is destroyed in general.

2.2 Type-2 symmetric structure

The type-2 symmetric structure of the solution set Σ of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$ described in this section comes only from the symmetry in the supports $\mathcal{A}_j^c$ $(j = 1, 2, \ldots, n)$, but not from the symmetry in the coefficients $c_j^c(\boldsymbol{a})$ $(\boldsymbol{a} \in \mathcal{A}_j^c,\ j = 1, 2, \ldots, n)$; hence even if we change some or all of the coefficients, the type-2 symmetric structure remains valid. From the description (1) of the supports $\mathcal{A}_j^c$ $(j = 1, 2, \ldots, n)$ of $\boldsymbol{f}^c(\boldsymbol{x})$, we see that, for every $j = 1, 2, \ldots, n-1$, all the terms in the jth component $f_j^c(\boldsymbol{x})$ have a common degree j, and that the nth polynomial $f_n^c(\boldsymbol{x})$ has two terms, one with the degree n and the other with the degree 0.

It follows that $\boldsymbol{x} \in \Sigma$ if and only if $\theta(n,k)\boldsymbol{x} \in \Sigma$ for any $k \in \{0,1,\ldots,n-1\}$. Here $\theta(n,k) = \exp(2\pi ki/n)$ for every positive integer n and every nonnegative integer $k \leq n-1$, and i denotes the imaginary unit. Thus one solution $\boldsymbol{x} \in \Sigma$ is expanded to n solutions $\theta(n,k)\boldsymbol{x} \in \Sigma$ $(k = 0,1,2,\ldots,n-1)$. If $\boldsymbol{x} \in \widetilde{\Sigma}$, such expanded solutions form an *orbit of type-2* of n nonsingular solutions of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$. Therefore, we can partition $\widetilde{\Sigma}$ into $2m(n)$ orbits of type-2, for example, $\widetilde{\Sigma}^2_1, \widetilde{\Sigma}^2_2, \ldots, \widetilde{\Sigma}^2_{2m(n)}$.

3 The polyhedral homotopy continuation method

Let $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x}))$ be a general polynomial system such that

$$\begin{aligned} f_j(\boldsymbol{x}) &= \sum_{\boldsymbol{a}\in\boldsymbol{A}_j} c_j(\boldsymbol{a})\boldsymbol{x}^{\boldsymbol{a}} \ (j=1,2,\ldots,n),\ \mathcal{A}_j \subset \mathbb{Z}^n_+ \ (j=1,2,\ldots,n) \ \text{ and} \\ c_j(\boldsymbol{a}) &\in \mathbb{C} \ (\boldsymbol{a} \in \mathcal{A}_j,\ j=1,2,\ldots,n). \end{aligned}$$

Throughout this section, we are concerned with the polynomial equations $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ to describe the polyhedral homotopy method that we use.

3.1 Outline of a standard homotopy continuation method

We begin by describing how we compute a single solution of $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ using a standard homotopy continuation method. The basic idea is to trace a smooth curve $\{(\boldsymbol{\xi}(t),t) \in \mathbb{C}^n \times \mathbb{R} : t \in [0,1]\}$ that connects a known $\boldsymbol{\xi}(0)$ to a solution $\boldsymbol{\xi}(1)$ of $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$. Here $\boldsymbol{\xi} : [0,1] \to \mathbb{C}^n$ is defined as a solution curve of a system of equations

$$\boldsymbol{h}(\boldsymbol{x},t) = \boldsymbol{0} \tag{3}$$

with a parameter $t \in [0,1]$; $\boldsymbol{h}(\boldsymbol{\xi}(t),t) = \boldsymbol{0}$ for every $t \in [0,1]$, and $\boldsymbol{h} : \mathbb{C}^n \times [0,1] \to \mathbb{C}^n$ is a smooth function such that

(a) all solutions of $\boldsymbol{h}(\boldsymbol{x},0) = \boldsymbol{0}$ are known or easily computed,

(b) $\boldsymbol{h}(\boldsymbol{x},1) = \boldsymbol{f}(\boldsymbol{x})$ for every $\boldsymbol{x} \in \mathbb{C}^n$, and

(c) each component h_j is a polynomial in variables $x_1, x_2, \ldots, x_n \in \mathbb{C}$ and $t \in [0,1]$.

We call $\boldsymbol{h}$ a homotopy function between $\boldsymbol{g}(\cdot) \equiv \boldsymbol{h}(\cdot,0) : \mathbb{C}^n \to \mathbb{C}^n$ and $\boldsymbol{f}(\cdot) : \mathbb{C}^n \to \mathbb{C}^n$, (3) a system of homotopy equations, and $\{(\boldsymbol{\xi}(t),t) : t \in [0,1]\}$ a homotopy curve.

To trace the homotopy curve $\{(\boldsymbol{\xi}(t),t) : t \in [0,1]\}$ numerically, we employ predictor and corrector procedures starting from a known solution $(\boldsymbol{\xi}(0),0)$ of (3). Let $\boldsymbol{x}^0 = \boldsymbol{\xi}(0)$ and $t^0 = 0$. Assume that a point $\boldsymbol{x}^k$ approximating $\boldsymbol{\xi}(t^k)$ for some $t^k \in [0,1)$ is computed at the kth iteration when $k \geq 1$ or given initially when $k = 0$.

In the predictor procedure, we compute an approximation $(\boldsymbol{dx},1)$ of the tangent vector $(\dot{\boldsymbol{\xi}}(t^k),1)$ of the curve $\{(\boldsymbol{\xi}(t),t) : t \in [0,1]\}$ at $t = t^k$ by solving a system of linear equation

$$\boldsymbol{D_x h}(\boldsymbol{x}^k,t^k)\boldsymbol{dx} = -\boldsymbol{D}_t\boldsymbol{h}(\boldsymbol{x}^k,t^k),$$

where $(\boldsymbol{D_x h}(\boldsymbol{x}^k,t^k), \boldsymbol{D}_t\boldsymbol{h}(\boldsymbol{x}^k,t^k))$ denotes the $n \times (n+1)$ Jacobian matrix of the homotopy function $\boldsymbol{h} : \mathbb{C}^n \times [0,1] \to \mathbb{C}^n$ at $(\boldsymbol{x},t) = (\boldsymbol{x}^k,t^k)$. After choosing a small step length

$\alpha^k > 0$ satisfying $t^{k+1} \equiv t^k + \alpha^k \leq 1$, we provide the first-order (rough) approximation $(\boldsymbol{x}^k, t^k) + \alpha^k(\boldsymbol{dx}, 1)$, of the solution $(\boldsymbol{\xi}(t^{k+1}), t^{k+1})$ of (3).

In the corrector procedure, we fix $t = t^{k+1}$ in (3) and apply the Newton method to the system of equations $\boldsymbol{h}(\boldsymbol{x}, t^{k+1}) = 0$ from the initial point $\boldsymbol{y}^0 = \boldsymbol{x}^k + \alpha^k \boldsymbol{dx}$. We continue to generate a sequence $\{\boldsymbol{y}^r\}$ until an approximate solution $\boldsymbol{y}^{r^*}$ of $\boldsymbol{h}(\boldsymbol{x}, t^{k+1}) = \boldsymbol{0}$ is attained with a prescribed accuracy. More precisely, each iteration of the corrector procedure is carried out by solving a system of linear equations $D_{\boldsymbol{x}}\boldsymbol{h}(\boldsymbol{y}^r, t^{k+1})\boldsymbol{dy} = -\boldsymbol{h}(\boldsymbol{y}^r, t^{k+1})$, and letting $\boldsymbol{y}^{r+1} = \boldsymbol{y}^r + \boldsymbol{dy}$. Let $\boldsymbol{x}^{k+1} = \boldsymbol{y}^{r^*}$. Replacing $k+1$ by k, we repeat the predictor and corrector procedures above until t^k becomes 1 or we obtain an approximation $\boldsymbol{x}^{k^*}$ of the solution $\boldsymbol{\xi}(1)$ of $\boldsymbol{h}(\boldsymbol{x}, 1) \equiv \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$.

To find all nonsingular solutions of $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$, we need one homotopy curve to reach each isolated nonsingular solution of $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$. Constructing such homotopy curves varies on the types of homotopy functions employed. Consider a homotopy function $\boldsymbol{h} : \mathbb{C}^n \times [0,1] \to \mathbb{C}^n$ satisfying the features (a), (b) and (c) above. Under a certain nondegenerate assumption, the connected component of $\{(\boldsymbol{x}, t) \in \mathbb{C} \times [0,1) : \boldsymbol{h}(\boldsymbol{x}, t) = \boldsymbol{0}\}$ containing each $(\boldsymbol{x}^0, 0) \in \{(\boldsymbol{x}, 0) : \boldsymbol{h}(\boldsymbol{x}, 0) = \boldsymbol{0}\}$ forms a 1-dimensional smooth curve of the form $\{(\boldsymbol{\xi}(t), t) : t \in [0,1)\}$. A homotopy curve behaves as one of the three cases: (i) $\boldsymbol{\xi}(t)$ converges to a nonsingular solution $\boldsymbol{x}^*$ of $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ as $t \to 1$. (ii) $\boldsymbol{\xi}(t)$ converges to a singular solution $\boldsymbol{x}^*$ of $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ as $t \to 1$. (iii) $\|\boldsymbol{\xi}(t)\| \to \infty$ as $t \to 1$. In case (i) and (ii), we obtain a solution $\boldsymbol{x}^*$ of $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ by tracing the homotopy curve, and, letting $\boldsymbol{\xi}(1) = \boldsymbol{x}^*$, we can extend the domain $[0,1)$ of the function $\boldsymbol{\xi}$ to the closed interval $[0,1]$. In case (iii), all the work of tracing the homotopy curve is wasted. The total number of homotopy curves to be traced is the sum of the number of homotopy curves to solutions of $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ and the number of "worthless" divergent homotopy curves. For the computational efficiency, we would like to choose homotopy functions that yield a small number of divergent homotopy curves, at the same time, produce as many homotopy curves as all nonsingular solutions of $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$.

3.2 The cheater's homotopy

We use the cheater's homotopy, a combination of the polyhedral and linear homotopies, which is originally proposed in [12]. See Section 5 of [13] for details of the cheater's homotopy. In the cheater's homotopy, we construct a class of homotopy functions $\boldsymbol{h}^p$ $(p = 1, 2, \ldots, p^*)$ for some finite number p^* from $\mathbb{C}^n \times [0,1]$ into $\mathbb{C}^n$ satisfying not only (a), (b) and (c) above but also the properties (d), (e) and (f) below:

(d) Each component $h_j^p(\boldsymbol{x}, t)$ of $\boldsymbol{h}^p(\boldsymbol{x}, t)$ is of the form

$$\sum_{\boldsymbol{a} \in \mathcal{A}_j} \left((1-t)\tilde{c}_j(\boldsymbol{a}) + tc_j(\boldsymbol{a})\right) \boldsymbol{x}^{\boldsymbol{a}} t^{\rho_j^p(\boldsymbol{a})} \tag{4}$$

$(j = 1, \ldots, n,\ p = 1, 2, \ldots, p^*)$. Here $\tilde{c}_j(\boldsymbol{a}) \in \mathbb{C}$ $(\boldsymbol{a} \in \mathcal{A}_j,\ j = 1, 2, \ldots, n)$ are randomly generated complex numbers and $\rho_j^p(\boldsymbol{a})$ $(\boldsymbol{a} \in \mathcal{A}_j,\ j = 1, 2, \ldots, n,\ p = 1, 2, \ldots, p^*)$ are nonnegative real numbers chosen according to the theory of the polyhedral homotopy continuation method [13] such that for each $j = 1, 2, \ldots, n$ and each $p = 1, 2, \ldots, p^*$, exactly two numbers of $\rho_j^p(\boldsymbol{a})$ $(\boldsymbol{a} \in \mathcal{A}_j)$ are zero and all others are positive.

(e) Each component $h_j^p(\boldsymbol{x}, 0)$ of $\boldsymbol{h}^p(\boldsymbol{x}, 0)$ is a binomial that we can easily compute all nonsingular solutions, say $\boldsymbol{x}^{pq}$ $(q = 1, \ldots, q_p)$ of $\boldsymbol{h}^p(\boldsymbol{x}, 0)$, where q_p is a positive number.

(f) For each nonsingular solution $\tilde{\boldsymbol{x}}$ of $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$, there is a unique $p \in \{1, 2, \ldots, p^*\}$ and a unique $q \in \{1, 2, \ldots, q_p\}$ such that $\tilde{\boldsymbol{x}}$ is connected to $\boldsymbol{x}^{pq}$ through a homotopy curve of $\boldsymbol{h}^p(\boldsymbol{x}, t) = \boldsymbol{0}, \ (\boldsymbol{x}, t) \in \mathbb{C}^n \times [0, 1]$.

Legitimate and efficient computation of the nonnegative numbers $\rho_j^p(\boldsymbol{a})$ $(\boldsymbol{a} \in \mathcal{A}_j, \ j = 1, 2, \ldots, n, \ p = 1, 2, \ldots, p^*)$ is based on the polyhedral combinatorics, and can be formulated as an interesting combinatorial optimization problems [4, 18].

4 An application of the cheater's homotopy to the cyclic-n polynomial equation

4.1 Utilizing the type-2 symmetric structure

Now we apply the cheater's homotopy to the cyclic-n polynomial equations $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$. Let $p \in \{1, 2, \ldots, p^*\}$ be fixed. Since the polynomial system $\boldsymbol{h}^p(\boldsymbol{x}, t)$ in $x_1, x_2, \ldots, x_n \in \mathbb{C}$ shares the same supports $\mathcal{A}_j^c$ $(j = 1, 2, \ldots, n)$ with $\boldsymbol{f}^c(\boldsymbol{x})$, the type-2 symmetric structure is preserved for the solution set of $\boldsymbol{h}^p(\boldsymbol{x}, t) = \boldsymbol{0}$ for every fixed $t \in [0, 1]$. To explore this type-2 symmetric structure, we define $S^p = \{(\boldsymbol{x}, t) \in \mathbb{C}^n \times [0, 1] : \boldsymbol{h}^p(\boldsymbol{x}, t) = \boldsymbol{0}\}$, and $S^p(t) = \{\boldsymbol{x} \in \mathbb{C}^n : \boldsymbol{h}^p(\boldsymbol{x}, t) = \boldsymbol{0}\}$ $(t \in [0, 1])$. Suppose that $t \in [0, 1]$. Then $\boldsymbol{x} \in S^p(t)$ iff $\theta(n, k)\boldsymbol{x} \in S^p(t)$ for every $k \in \{0, 1, \ldots, n-1\}$. Hence $U \subset \mathbb{C}^n \times [0, 1]$ is a connected component of S^p iff the set $\{(\theta(n, k)\boldsymbol{x}, t) : (\boldsymbol{x}, t) \in U\}$ is a connected component of S^p for every $k \in \{0, 1, \ldots, n-1\}$. We focus our attention on the connected components of S^p that intersect with the hyperplane $\mathbb{C}^n \times \{0\}$ at one of $(\boldsymbol{x}^{pq}, 0)$ $(q = 1, 2, \ldots, q_p)$. Then we can partition the collection Ξ^p of such connected components into a finite number of subcollections, say Ξ_r^p $(r = 1, 2, \ldots, r_p)$, for a finite number r_p, such that U and V belong to Ξ_r^p iff $V = \{(\theta(n, k)\boldsymbol{x}, t) : (\boldsymbol{x}, t) \in U\}$ for some $k \in \{0, 1, \ldots, n-1\}$. We assume that each $U \in \Xi_r^p$ forms a smooth curve $\{(\boldsymbol{\xi}(t), t) : t \in T\}$ for some smooth function $\boldsymbol{\xi} : T \to \mathbb{C}^n$, where T is either $[0, 1]$ or $[0, 1)$. More specifically, one of the following three cases occur as we mentioned in Section 3.1: (i) $T = [0, 1]$ and $\boldsymbol{\xi}(1)$ is a nonsingular solution of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$. (ii) $T = [0, 1]$ and $\boldsymbol{\xi}(1)$ is a singular solution of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$. (iii) $T = [0, 1)$ and $\|\boldsymbol{\xi}(t)\| \to \infty$ as $t \to 1$. Then each Ξ_r^p consists of exactly n homotopy curves. The total number q_p of homotopy curves in the collection Ξ^p is nr_p.

4.2 Computation of all nonsingular solutions of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$

In view of the discussions above, we need to trace only one of the homotopy curves in each Ξ_r^p. Therefore we can save the computation of $(n-1)$ homotopy curves among n homotopy curves in Ξ_r^p $(r = 1, 2, \ldots, r_p, \ p = 1, 2, \ldots, p^*)$. In case (i), $\boldsymbol{\xi}(1)$ belongs to an orbit of type-2, $\widetilde{\Sigma}_j^2$ of n nonsingular solutions of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$, and all other $n-1$ solutions in the orbit $\widetilde{\Sigma}_j^2$ are obtained from $\theta(n, k)\boldsymbol{\xi}(1)$ $(k = 1, 2, \ldots, n-1)$.

Theoretically, if we choose the coefficients $\tilde{c}_j(\boldsymbol{a})$ $(\boldsymbol{a} \in \mathcal{A}_j, \ j = 1, 2, \ldots, n)$ randomly, we may generically assume that the set $\{(\boldsymbol{x}, t) \in \mathbb{C}^n \times [0, 1) : \boldsymbol{h}^p(\boldsymbol{x}, t) = \boldsymbol{0}\}$ consists of a finite number of 1-dimensional curves. In practice, however, this never guarantees that each homotopy curve $U \in \Xi_r^p$ starting from some solution $\boldsymbol{x}^{pq}$ of $\boldsymbol{h}^p(\boldsymbol{x}, 0) = \boldsymbol{0}$ can be traced correctly; a jump to a different homotopy curve V of $\boldsymbol{h}^p(\boldsymbol{x}, t) = 0$ can occur while tracing the homotopy curve U.

Suppose that approximate nonsingular solutions $\hat{\boldsymbol{x}}^1, \hat{\boldsymbol{x}}^2, \ldots, \hat{\boldsymbol{x}}^s$ of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$ are obtained by tracing one of the homotopy curves in every Ξ^p_r ($r = 1, 2, \ldots, r_p, p = 1, 2, \ldots, p^*$) with the predictor and corrector procedures. Here $s \leq \sum_{p=1}^{p^*} r_p$ since some of the homotopy curves traced may converge to singular solutions or may diverge. Let $\widehat{\Sigma}^2_j = \{\theta(n,k)\hat{\boldsymbol{x}}^j \ (k = 0, 1, \ldots, n-1)\}$ ($j = 1, 2, \ldots, s$). We then

(A) check whether there exists a pair of $\hat{\boldsymbol{x}}^j$ and $\hat{\boldsymbol{x}}^k$ such that $\hat{\boldsymbol{x}}^k$ approximately belongs to the orbit $\widehat{\Sigma}^2_j$ induced from $\hat{\boldsymbol{x}}^j$.

If we detect such a pair of $\hat{\boldsymbol{x}}^j$ and $\hat{\boldsymbol{x}}^k$, we may conclude that either of $\hat{\boldsymbol{x}}^j$ or $\hat{\boldsymbol{x}}^k$ was computed incorrectly or there was a jump from one homotopy curve to another while generating either $\hat{\boldsymbol{x}}^j$ or $\hat{\boldsymbol{x}}^k$. This suggests to take more conservative (shorter) predictor steps and retrace the two homotopy curves. In the numerical experiments in the Section 6, (A) worked very effectively to retrieve some missing nonsingular solutions.

If no such a pair of $\hat{\boldsymbol{x}}^j$ and $\hat{\boldsymbol{x}}^k$ exists, each $\widehat{\Sigma}^2_j$ is likely to correspond to a different orbit of type-2. In this case, we

(B) classify all the generated approximate solutions $\bigcup_{j=1}^s \widehat{\Sigma}^2_j$ according to the type-1 symmetric structure into approximate orbits of type-1 $\widehat{\Sigma}^1_j$ ($j = 1, 2, \ldots, u$) for some positive integer u, and

(C) check whether each $\widehat{\Sigma}^1_j$ contains $2n$ distinct solutions of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$.

If all the computation up to (B) is correct, each approximate orbit $\widehat{\Sigma}^1_j$ corresponds to an orbit of type-1. However if the number of approximate solutions in $\widehat{\Sigma}^1_j$ is more than $2n$, computation of one of the solutions is wrong, or if the number is less than $2n$, then at least one nonsingular solution is lost while following the homotopy curves. Thus (C) can be served as an effective tool to check the correctness of the computation and find all nonsingular solutions of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$. When all $\widehat{\Sigma}^1_j$ ($j = 1, 2, \ldots, u$) satisfy (C), we conclude with certainty that $\bigcup_{j=1}^s \widehat{\Sigma}^2_j = \bigcup_{j=1}^u \widehat{\Sigma}^1_j$ approximate all the solutions of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$.

We will discuss detailed implementation of (A), (B) and (C) in Section 5.4.

5 Some implementation issues

5.1 Powers of the homotopy parameter t

Recall that each component $h^p_j(\boldsymbol{x}, t)$ of a cheater's homotopy function $\boldsymbol{h}^p(\boldsymbol{x}, t)$ is of the form (4) ($p = 1, 2, \ldots, p^*$). The term $((1-t)\tilde{c}_j(\boldsymbol{a}) + tc_j(\boldsymbol{a}))\, \boldsymbol{x}^{\boldsymbol{a}} t^{\rho^p_j(\boldsymbol{a})}$ drastically changes within a sufficiently small interval $[1-\epsilon, 1]$ when the power constant $\rho^p_j(\boldsymbol{a})$ is large. For example, nonzero $\rho^p_j(\boldsymbol{a})$ varies from 1.0 through 68109.5 in our numerical experiment on the cyclic polynomial with $n = 12$. In such cases, we need to take smaller predictor steps as the homotopy parameter t approaches to 1.0; hence we can expect a large number of predictor iterations. Therefore, constructing cheater's homotopy functions $\boldsymbol{h}^p(\boldsymbol{x}, t)$ ($p = 1, 2, \ldots, p^*$) with small power constants is essential to reduce the predictor iterations and hence the CPU time for tracing homotopy curves. More precisely, a choice of a vector $\boldsymbol{\omega}$ of *lifting constants* decides the power constants $\rho^p_j(\boldsymbol{a})$ ($\boldsymbol{a} \in \mathcal{A}_j, j = 1, 2, \ldots, n,\ p = 1, 2, ..., p^*$). Furthermore, a mixed-cell configuration is generated so that two $\rho^p_j(\boldsymbol{a})$ should be zero for each j and p. See

[14] for more details. We can always assume that nonzero $\rho^p_j(\boldsymbol{a})$ ($\boldsymbol{a} \in \mathcal{A}_j, j = 1, 2, \ldots, n$) is not less than 1 by scaling t as $t = s^\gamma$ in the homotopy parameter space if necessary.

To tackle the difficulty caused by large powers of t when tracing the homotopy curves, it is important to use a lifting vector $\boldsymbol{\omega}$ that determines better balanced powers. One approach in this direction can be found in [7]. It searches in the cone of all lifting vectors that induce the same mixed-cell configuration by solving a linear program (LP). An important feature of the LP is that the ratio of the number of constraints to the number of variables becomes extremely large as n grows (see Table 1). The cutting-plane method was used to successfully obtain the solutions of the LP. The effect of the power-balancing procedure for the cyclic-n ($n = 9, 10, \ldots, 12$) problems is shown in Table 1 (All LPs were solved by using ILOG CPLEX 7.0).

Table 1: The effect of the power-balancing procedure

cyclic-n problems	highest power of t		size of LP		
	before balancing	after balancing	number of variables	number of constraints	ratio
$n = 8$	5,415.4	330.5	58	23,306	401.8
$n = 9$	429,052.6	1,539.8	74	117,420	1,586.8
$n = 10$	223,478.3	4,202.2	92	422,962	4,597.4
$n = 11$	2,769,612.0	20,500.7	112	2,397,486	21,406.1
$n = 12$	6,470,588.2	68,108.5	134	6,562,542	48,974.2

5.2 Upper bounds for predictor step lengths

Let $p \in \{1, 2, \ldots, p^*\}$ be fixed throughout this subsection. Suppose that a component $h^p_j(\boldsymbol{x}, t)$ of the cheater's homotopy function $\boldsymbol{h}^p(\boldsymbol{x}, t)$ involves power constants $\rho^p_j(\cdot, \boldsymbol{a}) = p = 10$ and 100000. Then the corresponding t^p changes from 0.3 to 0.99 in the intervals $[0.99, 0.9999]$ and $[0.99999, 0.9999999]$, respectively. Thus when $h^p_j(\boldsymbol{x}, t)$ has power constants with various magnitudes, we need to take predictor step lengths so small that the corresponding terms that make most changes reside in the resulting intervals.

For every $t \in (0, 1)$, define $\psi(t) = \max\left\{ ds^{\rho^p_j(\boldsymbol{a})}/ds\Big|_{s=t} : \boldsymbol{a} \in \mathcal{A}_j, j = 1, 2, \ldots, n \right\}$. If the maximum is attained at $\boldsymbol{a} = \tilde{\boldsymbol{a}}$ and $j = \tilde{j}$ on the right hand side above, then the largest local change occurs in $s^{\rho^p_{\bar{j}}(\bar{\boldsymbol{a}})}$ among $s^{\rho^p_j(\boldsymbol{a})}$ ($\boldsymbol{a} \in \mathcal{A}_j, j = 1, 2, \ldots, n$) when s increases from the current value t slightly. Thus it is reasonable to take a predictor step length α satisfying $(t+\alpha)^{\rho^p_{\bar{j}}(\bar{\boldsymbol{a}})} - t^{\rho^p_{\bar{j}}(\bar{\boldsymbol{a}})} \leq \delta$, where $\delta > 0$ is a given small positive number; we took $\delta = 0.1$ as a default value in the numerical experiments in Section 6.

5.3 Divergent homotopy curves and singular solutions

As mentioned in Section 4.1, a homotopy curve may converge to a nonsingular solution as in case (i) , but it may converge to a singular solution as in case (ii) or it may diverge as in case (iii). Some methods [9, 16] were proposed to distinguish these three cases. Our method described below is rather simple mathematically, but it worked effectively in the numerical experiments. Let $\{(\boldsymbol{x}^k, t^k) \in \mathbb{C}^n \times [0,1]\}$ be a sequence generated by our homotopy continuation method. We assume that the sequence correctly follows a homotopy curve $\{(\boldsymbol{\xi}(t), t) : t \in T\}$ of the cheater's homotopy system $\boldsymbol{h}^p(\boldsymbol{x}, t) = \boldsymbol{0}$ in a high accuracy, where $T = [0, 1)$ or $[0, 1]$.

Let ϵ_1 be a sufficiently small positive number; for example, $\epsilon_1 = 1.0e\text{-}5$. Suppose that $1.0 - \epsilon_1 < t^k \leq 1.0$ holds at the kth iteration. In (iii), $\|\boldsymbol{\xi}(t)\|$ and $\|\dot{\boldsymbol{\xi}}(t)\|/(1.0-t)$ both diverge as $t \to 1.0$. Therefore we decide that the homotopy curve diverges if $1.0 - \epsilon_1 < t^k < 1.0$ and if $\|\boldsymbol{x}^k\|$ and $\|\boldsymbol{dx}\|/(1.0 - t^k)$ are larger than given threshold values, where $\boldsymbol{dx} \in \mathbb{C}^n$ is the predictor direction vector at $(\boldsymbol{x}, t) = (\boldsymbol{x}^k, t^k)$ obtained from $\boldsymbol{D_x h}(\boldsymbol{x}^k, t^k)\boldsymbol{dx} = -\boldsymbol{D}_t\boldsymbol{h}(\boldsymbol{x}^k, t^k)$. Otherwise we compute a solution of $\boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{h}(\boldsymbol{y}, 1) = \boldsymbol{0}$ by applying the Newton method from the initial point $\boldsymbol{y}^0 = \boldsymbol{x}^k$. We check the magnitudes of the Newton direction and the function value at each step,

$$\|\boldsymbol{D_y f}(\boldsymbol{y}^j)^{-1}\boldsymbol{f}(\boldsymbol{y}^j)\| < \epsilon_2 \text{ or } \|\boldsymbol{f}(\boldsymbol{y}^j)\| < \epsilon_3 \tag{5}$$

for some sufficiently small positive numbers ϵ_2 and ϵ_3; for example, $\epsilon_2 = 1.0e\text{-}7$ and $\epsilon_3 = 1.0e\text{-}8$. If one of these inequalities is satisfied at $j = \ell$, then $\hat{\boldsymbol{x}} = \boldsymbol{y}^\ell$ is regarded as an approximate solution. In this case, the investigation continues to check whether (I) the sequence $\{\det \boldsymbol{D_y f}(\boldsymbol{y}^j)\}$ of the determinants of the Jacobian matrices is bounded away from 0 and is expected to converge to a positive number as $j \to \infty$, or (II) it approaches to 0. We determine $\hat{\boldsymbol{x}}$ as a nonsingular solution in case (I) and a singular solution in case (II).

If $\{\boldsymbol{y}^j\}$ converges to a nonsingular solution then the convergence rate of $\|\boldsymbol{D_y f}(\boldsymbol{y}^j)^{-1}\boldsymbol{f}(\boldsymbol{y}^j)\|$ to 0 is quadratic, otherwise, it is not faster than linear. Assume that the Newton iterations end with $\|f(\boldsymbol{y}^\ell)\| < \epsilon_3$. Then, one way to examine whether $\hat{\boldsymbol{x}} = \boldsymbol{y}^\ell$ approximates a nonsingular or singular solution is to compare the magnitude of $\|\boldsymbol{D_y f}(\boldsymbol{y}^\ell)^{-1}\boldsymbol{f}(\boldsymbol{y}^\ell)\|$ with a small positive number, for example, $1.0e\text{-}6$; $\hat{\boldsymbol{x}} = \boldsymbol{y}^\ell$ is considered as an approximation of a nonsingular solution if $\|\boldsymbol{D_y f}(\boldsymbol{y}^\ell)^{-1}\boldsymbol{f}(\boldsymbol{y}^\ell)\|$ is smaller than the given number. We combined this strategy with the criteria (I) and (II) above in the numerical experiments.

If the generated iterate $\boldsymbol{y}^j$ is outside of a given small neighborhood about $\boldsymbol{y}^0$ or if it does not satisfy (5) in a given maximum number of iterations, we conclude that the homotopy curve diverges.

5.4 Numerical comparison of approximate solutions of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$

In this subsection, we discuss numerical methods for (A), (B) and (C) mentioned in Section 4.2. In all cases, we need to compare two approximate nonsingular solutions of the cyclic polynomial equations $\boldsymbol{f}^c(\boldsymbol{x})$ and determine whether they approximately belong to a common orbit of type-2 for case (A), to a common orbit of type-1 for case (B) or to a common nonsingular solution of $\boldsymbol{f}^c(\boldsymbol{x})$ for case (C). If the number of approximate nonsingular solutions was small, we could execute (A), (B) and (C) by comparing every pair of the solutions repeatedly. However such a primitive method never works effectively when the number of solutions becomes large; as we will see in Section 6, the cyclic-10, -11 and -12 problems have $34,940$, $184,756$ and $367,488$ nonsingular solutions, respectively. To resolve this difficulty, we introduce three types of real valued continuous functions on $\mathbb{C}^n$, κ_a, κ_b and κ_c for (A), (B) and (C), respectively. We use each function as a key function with which we sort approximate nonsingular solutions and compare the key function values of two consecutive approximate nonsingular solutions.

Ideally we want to have the property on κ_a such that for any pair of $\boldsymbol{x},\ \boldsymbol{x}' \in \widetilde{\Sigma}$,

$$\boldsymbol{x},\ \boldsymbol{x}' \in \widetilde{\Sigma}_j^2 \text{ if and only if } \kappa_a(\boldsymbol{x}) = \kappa_a(\boldsymbol{x}'). \tag{6}$$

Let $\hat{\boldsymbol{x}}^1, \hat{\boldsymbol{x}}^2, \ldots, \hat{\boldsymbol{x}}^s$ be approximate nonsingular solutions of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$, each obtained by tracing one of the homotopy curves in some Ξ_r^p ($r = 1, 2, \ldots, r_p, p = 1, 2, \ldots, p^*$). We sort the approximate nonsingular solutions $\hat{\boldsymbol{x}}^1, \hat{\boldsymbol{x}}^2, \ldots, \hat{\boldsymbol{x}}^s$ in the ascending order of the values of $\kappa_a(\hat{\boldsymbol{x}}^1), \kappa_a(\hat{\boldsymbol{x}}^2), \ldots, \kappa_a(\hat{\boldsymbol{x}}^s)$. If the approximate nonsingular solutions are accurate and $\epsilon_a > 0$ is small, $|\kappa_a(\hat{\boldsymbol{x}}^r) - \kappa_a(\hat{\boldsymbol{x}}^r)| \leq \epsilon_a$ is the criterion to conclude that two consecutive $\hat{\boldsymbol{x}}^q$ and $\hat{\boldsymbol{x}}^r$ are approximately in a common orbit of type-2. In the numerical experiments, we used the function $\kappa_a(\boldsymbol{x}) = \sum_{j=2}^n \gamma_j^1 \arg(x_j/x_1) + \sum_{j=1}^n \gamma_j^2 |x_j|$. Here γ_j^1 ($j = 2, 3, \ldots, n$), γ_j^2 ($j = 1, 2, \ldots, n$) are randomly generated real numbers in the interval $(0, 1)$, and $\arg(u) \in (-\pi, \pi]$ denotes the argument of a nonzero complex number u. Theoretically the "only if" part is guaranteed in (6), but this function worked quite effectively for the purpose of (A) in the numerical experiments except a few cases where $|\kappa_a(\hat{\boldsymbol{x}}^q) - \kappa_a(\hat{\boldsymbol{x}}^r)| \leq \epsilon_a$ holds for two consecutive $\hat{\boldsymbol{x}}^q$ and $\hat{\boldsymbol{x}}^r$ but they do not approximately belong to a common orbit of type-2. For those cases, if they satisfy $|\arg(\hat{x}_j^q/\hat{x}_j^r) - \arg(\hat{x}_1^q/\hat{x}_1^r)| < \epsilon_a'$ ($j = 2, 3, \ldots, n$), we decide that they approximately are in a common type-2 orbit, and otherwise in different orbits of type-2. Here ϵ_a' is a small positive number.

The numerical method for (B) is similar to the one discussed above for (A) except requiring the property for κ_b such that for any pair of $\boldsymbol{x},\ \boldsymbol{x}' \in \widetilde{\Sigma}$, $\boldsymbol{x},\ \boldsymbol{x}' \in \widetilde{\Sigma}_j^1$ if and only if $\kappa_b(\boldsymbol{x}) = \kappa_b(\boldsymbol{x}')$. It is not difficult to find real valued functions on $\mathbb{C}^n$ which satisfy the "only if" part; for example, $\sum_{j=1}^n \text{real}(x_j)\text{real}(x_{j+2})$, $\sum_{j=1}^n (|x_j||x_{j+3}|)$, and $\sum_{j=1}^n (\text{imag}(x_j)\text{real}(x_{j+4}) + \text{imag}(x_{j+4})\text{real}(x_j))$. Here the indices $j+2$, $j+3$ and $j+4$ should be replaced by $j+2-n$, $j+3-n$ and $j+4-n$ if they exceed n, and $\text{real}(x_j)$ and $\text{imag}(x_j)$ denote the real and the imaginary parts of x_j. In the numerical experiments, we used a linear combination κ_b of such functions with randomly generated coefficients; an effective linear combination was not easy to obtain but was constructed experimentally through trial and error. The approximate nonsingular solutions $\hat{\boldsymbol{x}}^1, \hat{\boldsymbol{x}}^2, \ldots, \hat{\boldsymbol{x}}^s$ are sorted in the ascending order of the values $\kappa_b(\hat{\boldsymbol{x}}^1), \kappa_b(\hat{\boldsymbol{x}}^2), \ldots, \kappa_b(\hat{\boldsymbol{x}}^s)$. Then we test $|\kappa_b(\hat{\boldsymbol{x}}^r) - \kappa_b(\hat{\boldsymbol{x}}^q)| \leq \epsilon_b$ for two consecutive $\hat{\boldsymbol{x}}^q$ and $\hat{\boldsymbol{x}}^r$ for a sufficiently small positive number ϵ_b to see that they are in a common orbit of type-1. This method was effective for (B) in the numerical experiments.

For (C), we introduced a function $\kappa_c(\boldsymbol{x}) = \displaystyle\sum_{j=1}^n \left((\beta_1)^j \text{real}(x_j) + (\beta_2)^j \text{imag}(x_j)\right)$, where we took $\beta_1 = 0.58$ and $\beta_2 = 0.60$. We then sorted the approximate nonsingular solutions $\boldsymbol{x}$'s in each $\widehat{\Sigma}_j^1$ ($j = 1, 2, \ldots, u$) in the ascending order of the key values $\kappa_c(\hat{\boldsymbol{x}})$'s, and checked if $|\kappa_c(\hat{\boldsymbol{x}}) - \kappa_c(\hat{\boldsymbol{x}}')| \geq \epsilon_c$ for two consecutive $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{x}}'$ in $\widehat{\Sigma}_j^1$, where ϵ_c is a sufficiently small positive number. If this inequality was satisfied, then we concluded that they corresponded to different nonsingular solutions of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$, otherwise we checked $\|\hat{\boldsymbol{x}} - \hat{\boldsymbol{x}}'\|$ to see whether they approximated a common solution of $\boldsymbol{f}^c(\boldsymbol{x}) = \boldsymbol{0}$. This method also worked well.

n	the number of variables
No.paths	the number of paths
MV	Mixed Volume
Av.pred.it.	the average number of predictor iterations per path.
Av.corr.it	the average number of corrector iterations per path.
Av.CPU	average CPU time per path.
No.orbits	the number of orbits.
No.nonde.sol	the number of nonsingular solutions counting multiplicity

Table 2: Notation

6 Numerical results

All the computation was implemented on Pentium3 800MHz with 1 GB memory. We use the notation in Table 2 in the discussion of the numerical experiments. Table 3 shows the statistics obtained while following the curves for the cyclic-8 to cyclic-12 polynomial problems. The size of nonsingular solution data of each problem is too big to be included in this paper. We refer the homepage [10] for actual solution values. No.paths indicates the number of paths that we followed using the type-2 symmetric structure as described in Section 4.2. We can see that the mixed volume in each problem is $n*$No.paths and the work of tracing the curves is reduced. Except the cyclic-9 problem, as n increases, the average number of predictor iterations grows, so do the average number of corrector iterations and cpu time. This is because the powers of the continuation parameter t become very large with the dimension, and tracing the curves encounters increasing numerical difficulties. We can use the average number of predictor iterations to compute the average sizes of steps taken during curve tracing. For instance, in the case of the cyclic-10 problem, the average predictor iterations 124.48 indicates average step size of 0.008. Cyclic-9 problem consumed more numbers of predictor and corrector iterations, cpu time than the cyclic-10 and 11 problems. Among 1224 curves traced, only 666 curves arrived to nonsingular solutions at $t = 1$ and about half of the curves either diverged or converged to singular solutions. Furthermore, the condition number of the Jacobian matrix during the course of curve tracing often became very large, which made it difficult to solve linear systems in the predictor and corrector procedures. This contributed to many retries in the predictor procedure and more iterations to converge to an approximate solution at t^k in the corrector procedure, resulting in small steps and large number of the predictor iterations.

To verify the results that we obtained for the dimension $n = 8, 10, 11$, we used Jan Verschelde's solution information of the cyclic problems available in [19] to compare our computation results. All nonsingular solutions that we have found are matched with the results in his homepage. The nonsingular solutions of the cyclic-9 and 12 polynomial problems are obtained with the cheater's homotopy continuation method successfully.

Most of the difficulties involving the cyclic-12 problem are originated from the huge number of curves, classifying the large number of solutions into 15312 orbits and the big powers of the continuation parameter t. The problem has 41,696 curves to follow. It is extremely time-consuming to deal with a single computer. Differentiating nonsingular solutions from singular solutions was also complicated, but the use of the strategy described

n	8	9	10	11	12
No.paths	320	1224	3594	16796	41696
MV	2560	11016	35940	184756	500352
Av.pred.it.	82.12	192.70	124.48	142.20	195.80
Av.co.it	182.23	514.33	288.25	345.51	468.27
Av.CPU	1.63	16.17	5.71	9.50	17.57
No.orbits	72	333	1747	8398	15312
No.nonde.sol	1152	5994	34940	184756	367488

Table 3: Numerical Experiments with C++ programs for Cyclic n Problems

in Section 5 made it possible to find all nonsingular solutions.

7 Concluding remarks

Reducing the powers of the continuation parameter t is crucial to achieve the numerical stability in tracing curves as discussed in Section 5.1. In Table 1, the highest power shows a sharp growth even with the power-balance, *e.g.,* the value increased from $20,500.2$ for $n = 11$ to $68,108.5$ for $n = 12$. We also computed for $n = 13$ and obtained the value $174,168$. This suggests that it is necessary to search for a good lifting vector $\boldsymbol{\omega}$ globally, namely, in the area of cones of all mixed-cell configurations. The problem can be formulated as a nonlinear combinatorial optimization problem. Currently we are developing heuristic methods to solve the problem efficiently.

A vital issue of (cheater's) homotopy continuation methods for computing all solutions of a polynomial system is reliability. As mentioned in Section 5.4, we were not guaranteed to reach an approximation of a solution $\boldsymbol{\xi}(1)$ of a polynomial system $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ by tracing a homotopy curve $\{(\boldsymbol{\xi}(t), t) : t \in [0, 1]\}$. If we carefully design a stable homotopy continuation method, however, the failure rate is very low. In all cases of our numerical experiments, the failure rates are less than 0.001. In general cases, two effective techniques exist to increase the reliability of homotopy continuation methods. If two approximates solutions obtained from tracing different homotopy curves are almost equal to each other in a reasonable accuracy then recompute both homotopy curves taking a smaller predictor step. The other technique is as follows. Given a polynomial system to be solved, prepare multiple different sets of homotopy functions with randomly generated coefficients (see (4) in the cheater's homotopy case). Then, trace the homotopy curves of each set to compute a set of approximate solutions. Thus we obtain multiple sets of approximate solutions of the polynomial system. Finally merge them into a set of approximate solutions. Even if a solution may be lost in one set, it is very unlikely that the same solution happens to be lost in all the other sets. Thus the reliability of the merged set should be increased considerably.

Although the current numerical experiments reported in the previous section were carried out in a single CPU, a significant feature of homotopy continuation methods for polynomial systems is that all homotopy curves can be computed simultaneously and independently in parallel. The authors have been working on a parallel implementation of the cheater's homotopy continuation method for solving larger polynomial systems.

We also applied the cheater's homotopy continuation method to economic-n polynomials with $n = 6$ through 14. The interested readers can access to the homepage [10] for the numerical results.

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