CMPSm : A **C**ontinuation **M**ethod for **P**olynomial **S**ystems (MATLAB version) — User's Manual

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Abstract. CMPSm is a MATLAB program using a predictor-corrector path-following method for tracing homotopy curves to approximate all isolated solutions of a system of polynomial equations. The program was originally developed for cheater's homotopy, but it can be also used for linear and polyhedral homotopies. The latest version CMPSm2 of this software, this manual and some numerical examples are available at http://www.is.titech.ac.jp/∼kojima/CMPSm/

Key words. Polynomial, Homotopy Continuation Methods, Polyhedral Homotopy, Cheater's Homotopy, Linear Homotopy.

1 Introduction

We consider a system of polynomial equations $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ from the n-dimensional complex space $\mathbb{C}^n$ into itself, where each $f_k(\boldsymbol{x}) \in \mathbb{C}$ of $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x})) \in \mathbb{C}^n$ denotes a polynomial in variables $x_1, x_2, \ldots, x_n \in \mathbb{C}$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ a variable vector in $\mathbb{C}^n$. We describe how concepts of polyhedral homotopy continuation methods are employed in CMPSm and the usage of CMPSm2, which is the latest version of CMPSm, with an examplary polynomial the cyclic-3 [3] throughout the paper:

$$1 - x_1x_2x_3 = 0,\ x_1x_2 + x_2x_3 + x_3x_1 = 0,\ x_1 + x_2 + x_3 = 0 \tag{1}$$

Homotopy continuation methods [1, 9, 11, 15, 16, 17, 18] are known as powerful numerical methods for computing all isolated solutions of $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$. A common strategy behind the methods is to prepare a homotopy (polynomial) function $\boldsymbol{h} : \mathbb{C}^n \times [0, 1] \to \mathbb{C}^n$ such that

(a) all solutions of *the starting polynomial system* $\boldsymbol{h}(\boldsymbol{x}, 0) = \boldsymbol{0}$ are easily attainable,

(b) *the target polynomial system* $\boldsymbol{h}(\boldsymbol{x}, 1) = \boldsymbol{0}$ coincides with $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$,

(c) for all t in $[0, 1)$, the system $\boldsymbol{h}(\boldsymbol{x}, t) = \boldsymbol{0}$ has only nonsingular solutions.

Then, starting from a known solution $\boldsymbol{x}^0$ of $\boldsymbol{h}(\boldsymbol{x}, 0) = \boldsymbol{0}$ with *the homotopy parameter* $t = 0$ and increasing the value of t, trace a solution curve of $\boldsymbol{h}(\boldsymbol{x}, t) = \boldsymbol{0}$ numerically in the space $\mathbb{C}^n \times [0, 1]$ to obtain a solution of the target system $\boldsymbol{h}(\boldsymbol{x}, 1) \equiv \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ at $t = 1$. Linear, polyhedral and cheater's homotopies represent popular homotopies. A MATLAB program CMPSm is based on a predictor-corrector path-following method for tracing solution curves of such homotopy systems. Some technical details of the method were presented in the paper [4]. Recently released software package PHoM [5] includes the whole procedure of a polyhedral homotopy continuation method from computation of fine mixed cells of a

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polynomial system to prepare a family of homotopy functions to tracing solution curves. It is implemented in C++.

The reliability of polyhedral homotopy continuation methods depends on high powers of the continuation parameter t and ill-conditioned Jacobian matrices in curve tracing. This issue becomes critical to the success of implementation of the polyhedral homotopy continuation methods as we face increasingly higher powers of the continuation parameters, and to the height that can be impossible to handle with available machine precision without special techniques. Related to this issue, we recently proposed modified homotopy functions with a new homotopy continuation parameter s and various scaling techniques to enhance the numerical stability. The modified homotopy functions were obtained using a change of the parameter t, $s = \log t$. CMPSm2 implements this modified homotopy with the new continuation parameter s and scaling techniques in predictor-corrector procedures of curve tracing, which represent major differences from the original verison CMPSm. For details on the modified homotopy functions, we refer to [7].

2 Homotopy systems

Let $\mathbb{R}$ and $\mathbb{Z}_+$ denote the set of real numbers and the set of nonnegative integers, respectively. For every $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ and $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_+$, we use the notation $\boldsymbol{x}^{\boldsymbol{a}}$ for the term $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. Then we can write each $f_k(\boldsymbol{x})$ of $\boldsymbol{f}(\boldsymbol{x})$ as $f_k(\boldsymbol{x}) = \sum_{\boldsymbol{a} \in \mathcal{A}_k} c_k(\boldsymbol{a}) \boldsymbol{x}^a$ for a finite subset $\mathcal{A}_k$ of $\mathbb{Z}^n_+$ and $c_k(\boldsymbol{a}) \in \mathbb{C}$ $(\boldsymbol{a} \in \mathcal{A}_k)$ $(k = 1, 2, \ldots, n)$. We call $\mathcal{A}_k$ the *support* of $f_k(\boldsymbol{x})$. For the cyclic-3 polynomial (1), we take

$$\left.\begin{array}{l}
\boldsymbol{a}_1^1 = (0,0,0), \ \boldsymbol{a}_1^2 = (1,1,1), \ \mathcal{A}_1 = \{\boldsymbol{a}_1^1, \boldsymbol{a}_1^2\}, \\
\boldsymbol{a}_2^1 = (1,1,0), \ \boldsymbol{a}_2^2 = (0,1,1), \ \boldsymbol{a}_2^3 = (1,0,1), \ \mathcal{A}_2 = \{\boldsymbol{a}_2^1, \boldsymbol{a}_2^2, \boldsymbol{a}_2^3\}, \\
\boldsymbol{a}_3^1 = (1,0,0), \ \boldsymbol{a}_3^2 = (0,1,0), \ \boldsymbol{a}_3^3 = (0,0,1), \ \mathcal{A}_3 = \{\boldsymbol{a}_3^1, \boldsymbol{a}_3^2, \boldsymbol{a}_3^3\}, \\
c_1(\boldsymbol{a}_1^1) = 1, \ c_1(\boldsymbol{a}_1^2) = -1, \ c_2(\boldsymbol{a}_2^1) = 1, \ c_2(\boldsymbol{a}_2^2) = 1, \ c_2(\boldsymbol{a}_2^3) = 1, \\
c_3(\boldsymbol{a}_3^1) = 1, \ c_3(\boldsymbol{a}_3^2) = 1, \ c_3(\boldsymbol{a}_3^3) = 1.
\end{array}\right\} \tag{2}$$

Let $\widetilde{\mathcal{A}}_k \supseteq \mathcal{A}_k$ and $c_k(\boldsymbol{a}) = 0$ $(\boldsymbol{a} \in \widetilde{\mathcal{A}}_k \backslash \mathcal{A}, \ k = 1, 2, \ldots, n)$. Each component $h_k(\boldsymbol{x}, t)$ of the homotopy whose zeros (*i.e.* homotopy solution curve) can be traced by CMPSm2 is written as

$$h_k(\boldsymbol{x}, t) = \sum_{\boldsymbol{a} \in \widetilde{\mathcal{A}}_k} \left((1-t)\tilde{c}_k(\boldsymbol{a}) + t c_k(\boldsymbol{a})\right) \boldsymbol{x}^{\boldsymbol{a}} t^{\rho_k(\boldsymbol{a})}. \tag{3}$$

Here $\tilde{c}_k(\boldsymbol{a}) \in \mathbb{C}$ and $0 \leq \rho_k(\boldsymbol{a}) \in \mathbb{R}$ are given numbers $(\boldsymbol{a} \in \widetilde{\mathcal{A}}_k)$. To avoid possible degeneracy while tracing homotopy solution curves, a common practice is to assign randomly generated complex numbers to all (or some) of $\tilde{c}_k(\boldsymbol{a})$'s. Obviously $\boldsymbol{h}(\boldsymbol{x}, 1) = \boldsymbol{f}(\boldsymbol{x})$ for every $\boldsymbol{x} \in \mathbb{C}^n$; hence (b) holds. We call $\rho_k(\boldsymbol{a})$ $(\boldsymbol{a} \in \widetilde{\mathcal{A}}_k, \ k = 1, 2, \ldots, n)$ *powers* of t. Given solutions $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^\ell$ of $\boldsymbol{h}(\boldsymbol{x}, 0) = \boldsymbol{0}$, and the data $\widetilde{\mathcal{A}}_k$, $c_k(\boldsymbol{a})$ $(\boldsymbol{a} \in \widetilde{\mathcal{A}}_k)$, $\tilde{c}_k(\boldsymbol{a})$ $(\boldsymbol{a} \in \widetilde{\mathcal{A}}_k)$ and $\rho_k(\boldsymbol{a})$ $(\boldsymbol{a} \in \widetilde{\mathcal{A}}_k)$ $(k = 1, 2, \ldots, n)$, we can apply CMPSm2 to trace homotopy solution curves of $\boldsymbol{h}(\boldsymbol{x}, t) = \boldsymbol{0}$ numerically, starting from $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^\ell$ at $t = 0$. We describe three special homotopies, a linear homotopy, a polyhedral homotopy and cheater's homotopy.

2.1 Linear homotopies

Let $\rho_k(\boldsymbol{a}) = 0$ ($\boldsymbol{a} \in \widetilde{\mathcal{A}}_k,\ k = 1, 2, \dots, n$). Then

$$h_k(\boldsymbol{x}, t) = \sum_{\boldsymbol{a} \in \widetilde{\mathcal{A}}_k} \left((1-t)\tilde{c}_k(\boldsymbol{a})\boldsymbol{x}^{\boldsymbol{a}} + tc_k(\boldsymbol{a})\boldsymbol{x}^{\boldsymbol{a}}\right) = (1-t)h_k(\boldsymbol{x}, t) + tf_k(\boldsymbol{x}).$$

Thus this leads to a linear homotopy $\boldsymbol{h}(\boldsymbol{x}, t) = (1-t)\boldsymbol{h}(\boldsymbol{x}, 0) + t\boldsymbol{f}(\boldsymbol{x})$. See the literatures [1, 9, 18] for more details of linear homotopies. The usual form of a component of linear homotopy function at $t = 0$ is $h_k(\boldsymbol{x}, 0) = x_k^{d_k} - r^{d_k}\exp(\alpha_k i)$ ($k = 1, 2, \dots, n$). Here $\mathbb{Z}_+ \ni d_k > 0$, $\mathbb{R} \ni r_k > 0$, $\alpha_k \in [0, 2\pi)$, and i denotes the imaginary unit. Then all the solutions of the starting system $\boldsymbol{h}(\boldsymbol{x}, 0) = \boldsymbol{0}$ are given by

$$\left(r_1 \exp\left(\frac{\alpha_1 + 2p_1\pi}{d_1} i\right), r_2 \exp\left(\frac{\alpha_2 + 2p_2\pi}{d_2} i\right), \dots, r_n \exp\left(\frac{\alpha_n + 2p_n\pi}{d_n} i\right)\right)$$

($p_k = 0, 1, \dots, d_k - 1,\ k = 1, 2, \dots, n$).

In addition to (2), let

$$\left.\begin{array}{l}
\boldsymbol{a}_1^3 = (4,0,0), c_1(\boldsymbol{a}_1^3) = 0, \boldsymbol{a}_2^4 = (0,3,0), c_2(\boldsymbol{a}_2^4) = 0, \boldsymbol{a}_2^5 = (0,0,0), c_2(\boldsymbol{a}_2^5) = 0,\\
\boldsymbol{a}_3^4 = (0,0,2), c_3(\boldsymbol{a}_3^4) = 0, \boldsymbol{a}_3^4 = (0,0,0), c_3(\boldsymbol{a}_3^5) = 0,\\
\mathcal{A}_1^L = \mathcal{A}_1 \cup \{\boldsymbol{a}_1^3\},\ \mathcal{A}_2^L = \mathcal{A}_2 \cup \{\boldsymbol{a}_2^4\},\ \mathcal{A}_3^L = \mathcal{A}_3 \cup \{\boldsymbol{a}_3^4\},\\
\tilde{c}_1^L(\boldsymbol{a}_1^1) = -r_1^4\exp(\alpha_1 i), \tilde{c}_1^L(\boldsymbol{a}_1^2) = 0, \tilde{c}_1^L(\boldsymbol{a}_1^3) = 1,\\
\tilde{c}_2^L(\boldsymbol{a}_2^1) = \tilde{c}_2^L(\boldsymbol{a}_2^2) = \tilde{c}_2^L(\boldsymbol{a}_2^3) = 0, \tilde{c}_2^L(\boldsymbol{a}_2^4) = 1, \tilde{c}_2^L(\boldsymbol{a}_2^5) = -r_2^3\exp(\alpha_2 i),\\
\tilde{c}_3^L(\boldsymbol{a}_3^1) = \tilde{c}_3^L(\boldsymbol{a}_3^2) = \tilde{c}_3^L(\boldsymbol{a}_3^3) = 0, \tilde{c}_3^L(\boldsymbol{a}_3^4) = 1, \tilde{c}_3^L(\boldsymbol{a}_3^5) = -r_3^2\exp(\alpha_3 i).
\end{array}\right\} \quad (4)$$

Then we have a linear homotopy system

$$h_k^L(\boldsymbol{x}, t) = \sum_{\boldsymbol{a} \in \mathcal{A}_k^L} \left((1-t)\tilde{c}_k^L(\boldsymbol{a}) + tc_k(\boldsymbol{a})\right)\boldsymbol{x}^{\boldsymbol{a}}\ (k = 1, 2, 3) \quad (5)$$

for the cyclic-3 polynomial (1). All data necessary to execute CMPSm2 for the linear homotopy (5) is in the directory 3cycL. Specifically, $\mathcal{A}_k^L$, $c_k(\boldsymbol{a})$ and $\tilde{c}_k^L(\boldsymbol{a})$ ($\boldsymbol{a} \in \mathcal{A}_k^L,\ k = 1, 2, 3$) are described in the *coefficient file* 3cycL/3cycL.coef as follows:

```
# 3cycL.coef
# Linear Homotopy
n = 3
m = 3 5 5
a1.1 = 0 0 0
a1.2 = 1 1 1
a1.3 = 4 0 0
   ...
a3.5 = 0 0 0
coef1.1r = 1
coef1.1i = 0
   ...
coef3.5i = 0
Rcoef1.1r = 1.0056838664e+00
   ...
```

```
Rcoef3.5i = -5.3778227550e-01
```

This file contains the six keywords "n = ", "m = ", "a$k.\ell$ = ", "coef$k.\ell$r = ", "coef$k.\ell$i = ", "Rcoef$k.\ell$r = ", and "Rcoef$k.\ell$i = ", where k and ℓ stand for positive integers. A single space is necessary before and after = in each key word. A line should start with one of the keywords to be taken as valid data by CMPSm2. Following two lines of comments in the beginning, the dimension 3 of the problem is specified by "n = 3", and then the cardinalities of $\mathcal{A}_1,\ \mathcal{A}_2,\ \mathcal{A}_3$ by "m = 3 5 5". It is followed by the description of the supports $\mathcal{A}_1,\ \mathcal{A}_2,\ \mathcal{A}_3$ of the cyclic-3 polynomial where each line denotes a$k.\ell\ =\ \boldsymbol{a}_k^\ell \in \mathcal{A}_k$ $(k = 1, 2, 3)$. Then each of the next several lines denotes

$$\text{coef}k.\ell\text{r} \;=\; \text{the real part of } c_k(\boldsymbol{a}_k^\ell) \;\text{ or coef}k.\ell\text{i} \;=\; \text{the imaginary part of } c_k(\boldsymbol{a}_k^\ell),$$

and each line of the last part means

$$\text{Rcoef}k.\ell\text{r} \;=\; \text{the real part of } \tilde{c}_k(\boldsymbol{a}_k^\ell) \;\text{ or Rcoef}k.\ell\text{i} \;=\; \text{the imaginary part of } \tilde{c}_k(\boldsymbol{a}_k^\ell),$$

respectively.

The solutions of the starting polynomial system $\boldsymbol{h}^L(\boldsymbol{x}, 0) = \boldsymbol{0}$ are described in a *cell file of initial points* cycL/3cycL:

```
# 3cycL
# Linear homotopy
NumOfSols = 24
# Root information
sol1 = 7.5150986885e-01
6.7150155847e-01
2.4789020526e-01
1.0110515289e+00
2.6675738920e-01
1.0079988358e+00
   ...
sol24 = 6.7150155847e-01
   ...
# Power information
=  0   0   0
=  0   0   0   0   0
=  0   0   0   0   0
```

This file includes the three key words "NumOfSols = ", "sol$k$ = ", and "= ", where k stands for positive integers. Any line starting without one of the key words is neglected. As specified by NumOfSols = 24, this file contains 24 solutions, sol1, sol2, ... sol24 of the starting polynomial system $\boldsymbol{h}^L(\boldsymbol{x}, 0) = \boldsymbol{0}$. The kth solution, sol$k = (x_1, x_2, x_3)$ is described such that

$$\begin{array}{l} \text{sol}k = \text{the real part of } x_1 \\ \text{the imaginary part of } x_1 \\ \dots \\ \text{the imaginary part of } x_3 \end{array}$$

Each line of the last part of the file provides the powers $\rho_k(\boldsymbol{a})$ $(\boldsymbol{a} \in \mathcal{A}_k^L)$ of the homotopy parameter t $(k = 1, 2, 3)$ that are zeros in the linear homotopy.

2.2 Polyhedral homotopies

Let $\widetilde{\mathcal{A}}_k = \mathcal{A}$ $(k = 1, 2, \ldots, n)$ and $\mathcal{A}_k$ denotes the support of $f_k(\boldsymbol{x})$. To compute all isolated solutions of a polynomial system $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$, we produce multiple polyhedral homotopy systems for an auxiliary polynomial system $\tilde{f}_k(\boldsymbol{x}) = \sum_{\boldsymbol{a} \in \mathcal{A}_k} \tilde{c}_k(\boldsymbol{a})\boldsymbol{x}^a = 0$ $(k = 1, 2, \ldots, n)$ with randomly generated coefficients $\tilde{c}_k(\boldsymbol{a})$ $(\boldsymbol{a} \in \mathcal{A}_k,\ k = 1, 2, \ldots, n)$ based on the polyhedral homotopy theory [2, 6, 11, 15, 16, 17, etc.], and then we construct a linear homotopy system between the auxiliary polynomial system with its solutions and the target polynomial system $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$. Here $\mathcal{A}_k$ denotes the support of $f_k(\boldsymbol{x})$ $(k = 1, 2, \ldots, n)$. See [11, 12, 14, 17] for numerical methods for constructing polyhedral homotopy systems. Therefore, for the case of the polyhedral homotopy, CMPSm2 is used twice to find the solutions of the target polynomial systems. First, we apply CMPSm2 to each polyhedral homotopy system $h_k^P(\boldsymbol{x}, t) = \sum_{\boldsymbol{a} \in \mathcal{A}_k} \tilde{c}_k(\boldsymbol{a})\boldsymbol{x}^a t^{\rho_k(\boldsymbol{a})} = 0$ $(k = 1, 2, \ldots, n)$, where the set of powers $\rho_k(\boldsymbol{a})$ $(\boldsymbol{a} \in \mathcal{A}_k,\ k = 1, 2, \ldots, n)$ differs from one polyhedral homotopy system to another but $\tilde{c}_k(\boldsymbol{a})$ $(\boldsymbol{a} \in \mathcal{A}_k,\ k = 1, 2, \ldots, n)$ remain the same for all polyhedral systems. Then, we apply CMPSm2 again to a linear homotopy system $\sum_{\boldsymbol{a} \in \mathcal{A}_k} \left((1-t)\tilde{c}_k(\boldsymbol{a}) + tc_k(\boldsymbol{a})\right)\boldsymbol{x}^a = 0$ $(k = 1, 2, \ldots, n)$ with the solutions of the auxiliary polynomial system.

The distinctive feature of a polyhedral homotopy system $\boldsymbol{h}^P(\boldsymbol{x}, t)$ is that exactly two of $\rho_k(\boldsymbol{a})$ $(\boldsymbol{a} \in \mathcal{A}_k)$ are zeros and all the others are positive for each $k = 1, 2, \ldots, n$. As a result, the starting polynomial system $\boldsymbol{h}^P(\boldsymbol{x}, 0) = \boldsymbol{0}$ is a binomial system, which we can easily find solutions. When applying CMPSm2, we need to scale the homotopy parameter t so that all positive $\rho_k(\boldsymbol{a})$'s are not less than 1. Such a scaling is always possible. In fact, replace the homotopy parameter t by $t = s/\rho^*$ where ρ^* denotes the minimum of all positive $\rho_k(\boldsymbol{a})$'s $(\boldsymbol{a} \in \mathcal{A}_k,\ k = 1, 2, \ldots, n)$. Then the modified homotopy $\hat{\boldsymbol{h}}(\boldsymbol{x}, s) = \boldsymbol{h}(\boldsymbol{x}, s/\rho^*)$ satisfies the requirement.

In the case of the cyclic-3 polynomial (1), we constructed two polyhedral homotopies based on the polyhedral homotopy theory; the one has the set of powers

$$\left.\begin{array}{l} \rho_1(\boldsymbol{a}_1^1) = \rho_1(\boldsymbol{a}_1^2) = 0,\ \rho_2(\boldsymbol{a}_2^1) = \rho_2(\boldsymbol{a}_2^2) = 0,\ \rho_2(\boldsymbol{a}_2^3) = 4.75143, \\ \rho_3(\boldsymbol{a}_3^1) = \rho_3(\boldsymbol{a}_3^2) = 0,\ \rho_3(\boldsymbol{a}_3^3) = 1.9309 \end{array}\right\} \tag{6}$$

stored with 3 starting solutions in a cell file of initial points 3cycP/3cycP1, and the other has the set of powers

$$\left.\begin{array}{l} \rho_1(\boldsymbol{a}_1^1) = \rho_1(\boldsymbol{a}_1^2) = 0,\ \rho_2(\boldsymbol{a}_2^1) = 1.9309,\ \rho_2(\boldsymbol{a}_2^2) = \rho_2(\boldsymbol{a}_2^3) = 0, \\ \rho_3(\boldsymbol{a}_3^1) = 0,\ \rho_3(\boldsymbol{a}_3^2) = 4.75143,\ \rho_3(\boldsymbol{a}_3^3) = 0 \end{array}\right\} \tag{7}$$

together with 3 starting solutions in another cell file of initial points 3cycP/3cycP2. All other data $\mathcal{A}_k$ $(k = 1, 2, 3)$, $c_k(\boldsymbol{a}) = \tilde{c}_k(\boldsymbol{a})$ $(\boldsymbol{a} \in \mathcal{A}_k,\ k = 1, 2, 3)$ and $\tilde{c}_k(\boldsymbol{a})$ $(\boldsymbol{a} \in \mathcal{A}_k,\ k = 1, 2, 3)$ are located in the coefficient file 3cycP/3cycP.coef. The structures of 3cycP/3cycP.coef and 3cycP/3cycP1 (and 3cycP/3cycP2) are the same as 3cycL/3cycL.coef and 3cycL/3cycL. Only small differences lie in the last parts of 3cycP/3cycP1 and 3cycP/3cycP2 where powers of the homotopy parameter t are described as follows:

```
# 3cycP1
# Power information
=  0  0
=  0  0  4.75143
=  0  0  1.9309
```

```
# 3cycP2
# Power information
=  0   0
=  1.9309   0   0
=  0   4.75143   0
```

2.3 Cheater's homotopies

We take $\widetilde{\mathcal{A}}_k = \mathcal{A}$ $(k = 1, 2, \ldots, n)$, where $\mathcal{A}_k$ denotes the support of $f_k(\boldsymbol{x})$. Then, cheater's homotopy which we use in CMPSm2 may be regarded as a combination of a linear and a polyhedral homotopy with randomly generated values $\tilde{c}_k(\boldsymbol{a})$ and $c_k(\boldsymbol{a})$ $(\boldsymbol{a} \in \mathcal{A}_k,\ k = 1, 2, \ldots, n)$. See [10] for more general and/or exact definition of the cheater's homotopy, and also [13] for coefficient-parameter polynomial continuation methods that include the cheater's homotopy as a special case. Exactly two of $\rho_k(\boldsymbol{a})$ $(\boldsymbol{a} \in \mathcal{A}_k)$ of cheater's homotopy are zeros and all the others are positive for each $k = 1, 2, \ldots, n$. This enables CMPSm2 to start from solutions of a binomial system with the coefficient $\tilde{c}_k(\boldsymbol{a})$'s at $t = 0$ and attain solutions of the target polynomial system at $t = 1$ by tracing homotopy solution curves. All data to implement cheater's homotopies for the cyclic-3 polynomial (1) are stored in the directory 3cycC. Specifically, the coefficient file 3cycC/3cyc.coef includes the data $c_k(\boldsymbol{a})$ and $\tilde{c}_k(\boldsymbol{a})$ $(\boldsymbol{a} \in \mathcal{A}_k,\ k = 1, 2, 3)$, while the cell files of initial points 3cycC/3cycC1 and 3cycC/3cycC2 contain the same powers of t and the same initial solutions as the files 3cycP/3cycP1 and 3cycP/3cycP2. As in the polyhedral homotopy, we need to scale the homotopy parameter t so that all positive $\rho_k(\boldsymbol{a})$'s are not less than 1.

3 Parameter file

A user can specify the parameters accINfVal, accInNewtonDir, beta, divMagOFx, dTauMax, NewtonDirMax and predItMax which control the behavior of CMPSm2. The below is the parameter file, para_3cycL.dat used when CMPSm2 is applied to the cyclic-3 polynomial (1) by a linear homotopy with the files 3cycL.coef and 3cycL mentioned in Section 2.1.

```
# para3cycL.dat
accINfVal= 1.e-10
accInNewtonDir= 1.e-8
# take beta = 1 for linear homotopy case
beta=1;
divMagOFx= 1.0e+3
dTauMax= 0.1
NewtonDirMax= 0.1
predItMax= 2000
```

Here "accINfVal=", "accInNewtonDir=", "beta=", "divMagOFx=", "dTauMax=", "NewtonDirMax=", "predItMax=" are keywords at the first column of a line, and lines begin with '♯' are comments which CMPSm2 neglects. "accINfVal" and "accInNewtonDir" are used as stopping criteria; when an approximate solution $\boldsymbol{x}$ satisfying either

$$\begin{cases} \max\{|f_k(\boldsymbol{x})| : k = 1, 2, \ldots\} \leq \text{accINfVal} \quad \text{and} \\ \|\boldsymbol{f}(\boldsymbol{x})^{-1}\boldsymbol{f}(\boldsymbol{x})\| \leq \text{accInNewtonDir} \end{cases}$$

or

$$\begin{cases} \max\{|f_k(\boldsymbol{x})| : k = 1,2,\ldots,n\} \leq \text{accINfVal} \ \text{ and} \\ \|\boldsymbol{f}(\boldsymbol{x})^{-1}\boldsymbol{f}(\boldsymbol{x})\| > \text{accInNewtonDir} \end{cases}$$

CMPSm2 decides that the homotopy curve has converged to a nonsingular solution in the former case, and a singular solution in the latter case. When the 2-norm of an iterate $\boldsymbol{x}$ becomes larger than divMagOFx or when the predictor iteration exceeds predItMax, CMPSm2 stops. The parameter "dTauMax" provides an upper bound for increases in $t^{\rho_k(\boldsymbol{a})}$ ($\boldsymbol{a} \in \mathcal{A}_k, k = 1,2,\ldots,n$) in the predictor procedure. If the 2-norm $\|d\boldsymbol{x}\|$ of the Newton direction is greater than the value NewtonDirMax$\times\|\boldsymbol{x}\|$, the corrector iteration is discarded and then a predictor iteration with a smaller step is retried. The smaller the values of these two parameters are, the more accurate CMPSm2 traces a homotopy curve, to avoid an illegal jump from the curve into another curve.

The parameter beta ≥ 1 serves as a scaling for the homotopy parameter t. However, it is not relevant in the linear homotopy since all powers $\rho_k(\boldsymbol{a})$ ($\boldsymbol{a} \in \mathcal{A}_k,\ k = 1,2,\ldots,n$) are zero. For the polyhedral or cheater's homotopy, we suggest to take

$$\begin{aligned} &1 \leq \ \text{beta} \ \approx \gamma \times \ \text{"the maximum of powers, } \rho_k(\boldsymbol{a})\text{'s over all homotopies"}, \\ &0.01 \leq \gamma \leq 0.1 \end{aligned}$$

for computational efficiency and numerical stability. When large dimensional problems are solved by the polyhedral or cheater's homotopy, the maximum of $\rho_k(\boldsymbol{a})$'s over all homotopies can be quite large, for example, it can exceed 200,000. Taking an appropriate beta improves the performance of CMPSm2 significantly for such cases. When the maximum is small, for example, less than 100, choose beta $= 1$.

Remark. Three additional parameters "informationLevel", "coeffSW" and "modSW" with the usages are embedded in the main program CMPSm2.m.

4 Execution of CMPSm2

We briefly explain the procedure to solve the cyclic-3 polynomial using the linear homotopy (5) with (2) and (4):

(i) Place the following files in a directory:

- CMPSm2.m, traceOneCurve2.m — MATLAB program files,
- para3cycL.dat — a parameter file,
- 3cycL.coef, 3cycL — coefficient and cell files of initial points.

(ii) Run MATLAB.

(iii) Specify three input arguments for CMPSm2 in the MATLAB environment as

```
>>parameterFile='para3cycL.dat';
>>coefficientFile='3cycL.coef';
>>cellFileInitPt='3cycL';
```

(iv) Execute CMPSm2 as

```
>>CMPSm2(parameterFile,coefficientFile,cellFileInitPt);
```

Alternatively, combining (iii) and (iv) above, we can execute

```
>>CMPSm2('para3cycL.dat','3cycL.coef','3cycL');
```

When we use the polyhedral or cheater's homotopy, we need to specify several "cellFileInitPt"s for input arguments. CMPSm2 can accept two more input arguments, "startCellNo" and "endCellNo" to handle those cases. For solutions of the cyclic-3 problem by cheater's homotopy with the data files 3cycC.coef, 3cycC1, 3cycC2 and a parameter file para3cycC.dat, issue a command as

```
>>CMPSm2('para3cycC.dat','3cycC.coef','3cycC',1,2);
```

Here we take startCellNo= 1 and endCellNo= 2.

5 Output files

CMPSm2 generates two output file, *.stat file and *.sol file. Here * stands for the file name "cellFileInitPt" specified as the third input argument. The following is an example of the *.stat file from the cyclic-9 polynomial stored in the directory 9cycC.

```
# 9cyc1-978.stat
#  cell  init statusP pIT  TcIT  cpu  normOFx hValError normOFdx  minSing
   ...
      9    10  +3      86   164 2.67 4.98e+00  1.71e-14 4.99e-15 1.38e+00
   ...
     23     1  +4      83   245 3.18 4.90e+00  1.11e-13 2.10e-07 2.04e-07
   ...
     28     1  -2     171   410 5.92 1.02e+06  8.66e+00 7.03e+05 9.56e-12
```

Here "cell" denotes a positive integer attached to a cell file of initial points, which takes a value from "startCell" to "endCell" (the last two input arguments). "init" means the initial solution number. "statusP" indicates whether the homotopy path converged to a nonsingular solution (statusP = 3), converged to a singular solutions (statusP = 4) or diverges (statusP = -2). Also statusP can have other values in different circumstances. The meaning of the values is described at the top of the *.stat file. "pIT", "TcIT" and "cpu " stand for the total number of predictor iterations, the total number of corrector iterations and the cpu time spent to trace the homotopy path, respectively. "normOFx", "hValError" and "minSing" denote the 2-norm of an approximate solution $\boldsymbol{x}$ computed, the 1-norm of errors in function values of $\boldsymbol{x}$, and the minimum singular value of $f(\boldsymbol{x})$, respectively. These values are meaningful only when statusP is either +3, +4 or +5.

The other output file is a solution file. For the cyclic-9 polynomial problem, we have

```
# 9cyc1-978.sol
+7.6604444311e-01 . . . +9.3781448372e-02  +4.1789657759e-15   9 10
   ...
+1.7364863502e-01 . . . -3.7616321836e-01  +2.1321577253e-12  23  1
```

Each line without comment mark # shows an approximate solution $\boldsymbol{x}$, the 1-norm of errors in function values, the cell number of initial points and the initial solution number, where the last three numbers correspond "hValError', "cell" and "init" in the *.stat file, such that

$$\mathrm{real}(x_1), \mathrm{imag}(x_1), \ldots, \mathrm{real}(x_n), \mathrm{imag}(x_n), \mathrm{hValError}, \mathrm{cell}, \mathrm{init}.$$

Table 1: Numerical results from economic n problems by cheater's homotopy

n	6	7	8	9	10
No.paths	16	32	64	128	256
Av.pred.it.	120.2	103.8	108.1	118.0	142.0
Av.corr.it.	215.6	197.0	208.3	234.3	291.1
Av.CPU	1.9	2.0	2.5	3.1	4.6
No.sol.	16	32	64	128	256

6 Numerical results

Table 6 shows numerical results on the economic polynomial with dimensions 6, 7, ..., 10 solved by cheater's homotopy with the data stored in the directories 6ecoC, 7ecoC, ..., 10ecoC. The computation was done using MATLAB 6.5 Release 13 and Power Mac G4 1.0GHz. Here the following notation is used:

n	: the number of variables.
No.paths	: the number of paths traced
Av.pred.it.	: the average number of predictor iterations per path.
Av.corr.it.	: the average number of corrector iterations per path.
Av.CPU	: the average CPU time per path.
No.sol.	: the number of (nonsingular) solutions computed.

7 Concluding remarks

CMPSm2 may not be robust to find all solutions of a polynomial system for some cases, although the rate of missing a solution is very low, less than 0.1% from our experience. Two effective techniques exist to recover missing solutions. The one is to recompute the homotopy curves that have converged to a common solution with more conservative parameters ; take dTauMax= 0.05 and NewtonDirMax= 0.01 instead of the default values 0.5 and 0.1. The other is to apply CMPSm2 to more than a set of homotopy functions for a polynomial system to be solved, and merge multiple sets of the solutions obtained into a set of the solutions; choosing a different $\beta \geq 1$ in the parameter file and/or resetting coeffSW to -1 in CMPSm2 would yield different homotopies. See [4] for more details.

Main changes from CMPSm to CMPSm2 are that CMPSm2 implements the new modified homotopy functions with the continuation parameter s, and scaling techniques for solving linear systems with the Jacobian matrix in predictor-corrector procedure of curve tracing. We have observed from numerical experiments [7] that these new changes contribute to improve numerical stability.

An important advantage of homotopy continuation methods lies in parallel computation. Indeed, we can trace all homotopy paths simultaneously in parallel if the input data are split in a consistent manner. CMPSm2 was designed to benefit from this feature. For example, if three MATLAB environments are set up in three different cpus, we can execute each of the following commands

```
>>CMPSm2('para9cycC.dat','9cycC.coef','9cycC',1,300);
>>CMPSm2('para9cycC.dat','9cycC.coef','9cycC',301,600);
>>CMPSm2('para9cycC.dat','9cycC.coef','9cycC',601,978);
```

in one of three MATLAB environments, instead of issuing a command

```
>>CMPSm2('para9cycC.dat','9cycC.coef','9cycC',1,978);
```

in a single MATLAB environment to solve the economic problem of the dimension 9 whose data is stored in 9cycC.

The original version CMPSm and the current version CMPSm2 have been developed as a part of a joint project [4, 14] of (parallel) implementation of polyhedral homotopy continuation methods. A partial outcome of this project was reported in [4] with some numerical results on the cyclic polynomial system of the dimensions $8, 9, \ldots, 12$, which was obtained by a C++ code implementing cheater's homotopy continuation method. They also succeeded in approximating all isolated solutions of the cyclic-13 polynomial system by a parallel implementation of the C++ code. It resulted in 2,704,156 isolated solutions (counting multiplicity). See [8]. The original version CMPSm served as a prototype for the C++ code used. But CMPSm2 itself is not as fast as the C++ code for larger dimensional polynomial systems. The software package PHoM that contains the C++ version of CMPSm was released recently. It integrated the whole procedures for implementing polyhedral homotopy continuation methods from constructing of a family of polyhedral-linear homotopy functions to tracing solution paths.

Acknowledgment. Most of input data for cyclic and economic polynomials in this paper and the homepage http://www.is.titech.ac.jp/∼kojima/CMPSm was constructed using Professor Dai's and Dr. Takeda's C++ programs. The authors would like to express appreciation for their help.

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